Lectures on Special Relativity

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by

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PREFACE

"The special relativity theory, which was simply a systematic extension of the electrodynamics of Maxwell and Lorentz, had consequences which reached beyond itself ..."

Einstein, writing in *The Times*, November 28 1919.

Imagine a large number of starships, containing well found laboratories and competent physicists, moving along different straight lines at different constant speeds. Experiments can be carried out on board each ship, and astrophysical observations may be made from each ship. In each and every ship scientific instruments work in the same way, standard laboratory experiments give the same results and from both laboratory experiments and astrophysical observations everyone infers the same fundamental laws of physics ...the swarm of starships exists in a universe governed by a principle of relativity.

Look at some individual experiments. Interstellar analogues of Galileo and Newton play with billiard balls in their laboratories and find them to be governed by Newtonian mechanics. Analogues of Huyghens, Fresnel and Young study optics, analogues of Coulomb, Ampere and Faraday study electricity and magnetism. Analogue Maxwells synthesise the Maxwell equations. In every ship the same physical laws are found, and in particular every analogue Michelson discovers that the speed of light is independent of its direction, and it has the same measured value in every ship. Now there is a problem, because the invariance of the speed of light is at odds with the vector addition of velocities worked out by the mechanics.

The universe is so constructed that the speed of light is indeed measured to be the same, in all directions, regardless of which starship the measurement is made in. The vector addition of velocities worked out using billiard balls or relatively low velocity excursion modules is in fact wrong, and measured velocities do not compound in that way as any velocity becomes significant in comparison with that of light. This is all experimental fact and at first sight seems incomprehensible.

Incomprehension lifts (to some extent) with the realisation that measurements are made with instruments, instruments are made of matter, and matter is complicated ...so complicated that a measuring stick moving very fast relative to any one of our starships is contracted along the line of relative motion, a clock moving very fast runs slow, and mass is an increasing function of velocity. All these features are inherent in classical electromagnetism, and must be shared by all forms of matter in a universe governed by a principle of relativity. Such a principle places stringent restrictions on the form of physical law and this is the matter of the special theory of relativity.

Special relativity is now 80 years old and is diffusing into the early stages of university physics courses. There is a feeling abroad that exciting things like relativity should be taught as early as possible, the implication being that the physics of the 19th century is not exciting. This is unfortunate, for the physics of the 19th century is enormously impressive and exciting and much of it is novel to the student commencing university work. Special relativity only just avoided the 19th century and is the daughter of classical electromagnetic theory, which was formulated only some 40 years earlier.

An early introduction to special relativity has disadvantages. One is that for lack of appropriate mathematical tools rather arid problems in kinematics assume a disproportionate importance. Another is that familiarity breeds contempt, and many will conclude that relativity, too, is dull. But to my mind the greatest disadvantage is that an early course in relativity usually precedes study of electromagnetism and the student is thereby deprived of a background which makes sense of the apparently nonsensical.

A number of years ago I lectured to second year undergraduates at Oxford on the subject of special relativity, but it was only recently that I was goaded (by the perversity of some of my colleagues) to write up that material for publication. My aim when preparing the lectures which appear in this book was to give a concise account of the essential content of special relativity, without compromising the development of the subject by avoiding (relatively) advanced mathematics. Since the lectures were designed for students who knew little or no relativity, I was also concerned to pay particular attention to those difficulties, conceptual rather than mathematical, which invariably snare the vast majority.

Lecture 1 is concerned with the nature of a principle of relativity, the definition of inertial frames and the relationship between coordinates measured in different frames. The conflict between Newtonian mechanics, electromagnetism and a universal principle of relativity is spelt out and illustrated, and the lecture ends with a demonstration that electromagnetic matter will suffer Lorentz contraction. I believe that the realisation that classical electromagnetism predicts that electromagnetic measuring rods are shrunk and electromagnetic clocks are slowed when moving relative to the putative aether is of great help in understanding that the Lorentz transformations are physically perfectly acceptable. Lecture 2 is devoted to a careful and rigorous derivation of the Lorentz transformations themselves, with particular reference to their reciprocal nature. Lecture 3 deals with the related phenomena of time dilation and Lorentz contraction. Here I have been concerned to make clear which way round the formulae should be worked, the reciprocal nature of each phenomenon, and the absence of paradox.

Lecture 4 is largely devoted to the mathematical techniques which are necessary for grasping the essence of special relativity and which make so much easier many calculations where they cannot be held essential. These techniques are then put to work in obtaining the relativistic Doppler shift and in studying the covariance of Maxwell's equations. The requirement that conservation of energy and momentum be covariant leads in Lecture 5 to the identification of the correct expressions for energy and momentum as components of a 4-vector. The famous relation $E = mc^2$ is extracted and its physical significance is discussed. 4-forces are introduced and the relations between force, rate of change of momentum and acceleration are developed. The whole works is illustrated by writing the Lorentz force law in manifestly covariant terms.

Special relativity is verified continually in any high energy physics laboratory and the manipulation of relativistic kinematics is a tool of the trade of the high energy physicist. Lecture 6 is concerned wholly with the tricks of this trade and is liberally illustrated with real examples and problems drawn from high energy physics. It is for the reader who wishes to become familiar with this practical application of relativity and may be skimmed or skipped by those who find such application repellent.

Lecture 7 returns to the subject of the correct equations of motion of particles experiencing a force and deals briefly with the treatment of physical laws obtaining in an accelerated frame of reference—one of our starships when the drive is on. The interminable problem known as the twin paradox is then treated in considerable detail—80 years after the genesis of special relativity this problem continues to perplex successive generations of students, and not a few of their seniors.

Lecture 8 is to a large extent disjoint from the rest of the book. In the context of special relativity, neither matter, energy nor information can be propagated faster than light. But there are lots of things which go faster than light and in the absence of a careful analysis the consistency of special relativity as a covering principle for the physical world may be questioned. The lecture consists of a number of examples of things which go faster than light, ranging from the homely example of scissors through the old problems of phase and group velocity in classical physics to examples drawn from astrophysics, such as the apparent superluminal expansion of certain quasars. The conceptual problems encountered in arranging a marriage between relativity and quantum mechanics are discussed.

It would be best if the reader of this book were already acquainted with electromagnetism up to the level of Maxwell's equations and waves in empty space. I hope however that the book is sufficiently self explanatory that those whose studies are not that far advanced will nonetheless be able to acquire an understanding of the principle of relativity and the marvellous construction of the physical world which is expressed therein.

As in my previous lecture note volume, *Lectures on Statistical Mechanics* (Pergamon, 1982), PROBLEMS are scattered liberally throughout the text. They should be regarded as an integral part of the course. Many are simple exercises designed to further understanding of the fundamental material, while others are there to inculcate facility in solving realistic problems. The vast majority are very quick and easy ... but not all.

CONTENTS

Lecture 1 WHAT IS RELATIVITY?

A particular physical system satisfies a principle of relativity if the laws governing that system take the same form in all frames of reference moving with constant velocity. Thus formulated, relativity is a property of a particular set of physical theories. If all laws of physics take the same form, then the principle of relativity becomes rather a conceptual framework which must be satisfied by any particular theory, and a fundamental property of the physical universe.

Special relativity (Einstein's relativity) provides such a framework, into which the whole of physics seems to fit and which provides general principles within which any new theoretical model should be constructed. This framework has superseded that of Galileo and Newton (which was very fruitful and is very accurate provided only relatively small velocities are involved), which in turn superseded that of Aristotle (essentially useless). Relativistic mechanics (the theory of mechanics which satisfies the principles contained within special relativity) is no more difficult than Newtonian mechanics: electrodynamics as formulated by Maxwell satisfied these principles from the outset.

Consider a physicist studying a particular physical system. The simplest system he could study is a single particle not acted on by any force. It is an abstraction from experience that such a particle moves in a straight line at constant speed, when measured with linearly calibrated instruments, unless the physicist and his instruments are themselves accelerated. A frame of reference in which such a free test particle moves uniformly is called an *inertial frame* of reference.

A second observer, moving with his laboratory paraphenalia with constant velocity relative to the first, will see the single free test particle move with a different constant velocity. (The simplest situation is a particle at rest with respect to the apparatus of one observer, but moving with constant velocity with respect to the other). These considerations serve to define the set of inertial frames of reference.

We now let our observers watch the behaviour of a more complicated physical system: billiard balls on elastic strings, an atom emitting radiation, a star cluster, or whatever you like. If a universal principle of relativity holds, any physicist in any inertial frame will deduce the same laws of physics: the same in form and the same in numerical content; that is, the laws of physics assume the same form in all inertial frames of reference.

However philosophically attractive such a principle may seem, its applicability to the real world must be tested by investigating the real world (and remember that in accelerated frames of reference the laws of physics seem to assume a different form).

Newtonian mechanics embodies a principle of relativity — not Einstein's relativity but Galileo's.

Consider Newton's laws:

1. 'A body will move in a straight line at constant speed so long as no external force acts on it'.

(This is not circular: it is implicit that something must be around to produce a force).

We had better add explicitly the qualification 'observed from an inertial frame, measurements being made with linearly calibrated instruments' and assume that we really can make such measurements. This law may be taken as defining the family of inertial frames.

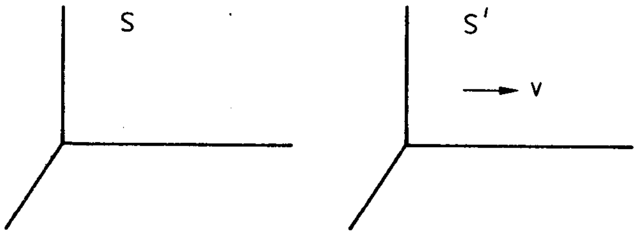
2. Force = mass × acceleration

which may be expressed in any of the forms

$$\mathbf{F} = m\mathbf{a} \qquad \text{or} \qquad \mathbf{F} = m\frac{d^2\mathbf{x}}{dt^2} \qquad \text{or} \qquad F_i = m\frac{d^2x_i}{dt^2}$$

F and **a** (and **x**) are vectors: this equation retains its form and numerical content under both rotations and translations of the coordinate system. The quantities x_i are the three orthogonal spatial coordinates and t (time) is a universal parameter which can be eliminated to yield the trajectory (as opposed to the equation of motion) of an object (for example the parabola described by a flung brick or the ellipse traced by a planet).

How are the coordinates of a particle in one frame related to the coordinates of the same particle as measured in another (inertial) frame? Let the two frames move with relative velocity v along common x axes



A marker at fixed x' in S' moves with velocity v along the x axis in S, such that

$$\left.\frac{dx}{dt}\right|_{S'} = v \tag{1.1}$$

and conversely

$$\left.\frac{dx'}{dt'}\right|_{S} = -v \tag{1.2}$$

These relations constitute the definition of relative velocity. The Galilean relation between x and x' is (choosing coincident origins for convenience)

$$\begin{aligned} x' &= x - vt \qquad & t' &= t \qquad \text{or} \\ x &= x' + vt' \qquad & t &= t' \end{aligned} \tag{1.3}$$

These are consistent equations which satisfy (1.1) and (1.2), but they are NOT the ONLY equations satisfying (1.1) and (1.2).

[PROBLEM: Find another set]

Suppose that in S'

$$F'_{x'} = m\frac{d^2x'}{dt'^2}$$

The Galilean transformation then yields

$$\begin{aligned} \frac{dx'}{dt'} &= \frac{dx'}{dt} = \frac{dx}{dt} - v \qquad \text{or} \qquad u'_{x'} = u_x - v \\ \frac{d^2x'}{dt'^2} &= \frac{d^2x}{dt^2} \end{aligned}$$

and so if $\mathbf{F}$ and m are the same in all such inertial frames the Galilean transformation takes you from Newton's laws in one frame to Newton's laws in another — same form and same numerical content. Newton's laws are *covariant* with respect to the Galilean transformations. Look into the origin of force a little further and introduce a potential:

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x} \qquad \text{so} \qquad \nabla'\phi' = \nabla\phi$$

and

$$m\frac{d^2\mathbf{x}'}{dt'^2} = -\nabla'\phi' \qquad \text{transforms into}$$

$$m\frac{d^2\mathbf{x}}{dt^2} = -\nabla\phi$$

and the equation holds equally well for all observers in inertial frames, provided that the scalar potential ϕ has the same value at the location of the particle in all frames; that is, ϕ is invariant under the Galilean transformations.

The corollary of all this is that there is no way of using Newtonian mechanics to define a meaningful absolute velocity — but if this seems obvious, contrast the case of accelerations.

The relation

$$\mathbf{u}' = \mathbf{u} - \mathbf{v}$$

may seem obvious. It isn't. Many people have terrible difficulties with problems involving vector addition of velocities at the first encounter. Obvious or not, it works — at low velocities.

Let's go on to another part of antediluvian physics — optics and electromagnetism as amalgamated by Maxwell. Maxwell's equations give the velocity of light equal to c (see Appendix) and we at once ask: with respect to what?

Suppose we suggest half an answer: with respect to some particular inertial frame (the aether frame). Then we expect that in some other inertial frame that a pulse of light would propagate with velocity $\mathbf{c}' = \mathbf{c} - \mathbf{v}$ and Maxwell's wonderful equations would not be true in any other frame. Why worry? After all, the wave equation for sound gives the velocity with respect to the medium supporting it and this is Newtonian physics. Surely we can keep a principle of relativity in the same sense for electromagnetism? Philosophically yes, but such a principle would be barren for light pervades the entire universe and would seem to define a universal fancy reference frame defined by Maxwell's equations being true in that frame only.

Let's look at the troubles with light a bit more quantitatively. The wave

$$\mathbf{E} = \mathbf{E}_0 \sin(\mathbf{k}.\mathbf{x} - \omega t) \qquad k^2 = \frac{\omega^2}{c^2} \tag{1.4}$$

and its associated magnetic field satisfy Maxwell's equations in empty space. The argument of the oscillatory function is the phase of the wave and is a pure number (proportional to a number of wave crests). It may be visualised as the number of wave crests passing the point $\mathbf{x}$ (or $\mathbf{x}'$) between the time of arrival of the crest which was at $\mathbf{x} = 0$, $t = 0$ and the time t, multiplied by (-2π). Since the origins of S, S' were defined to coincide and $\mathbf{x}'$ at $t' = t$ refers to the same point in space as $\mathbf{x}$ at the time t, the counting operation will yield the same number in any inertial frame and the phase

$$\phi = \mathbf{k}.\mathbf{x} - \omega t$$

will thus be an invariant.

[PROBLEM: Convince yourself of this. Remember that a wave crest passing the origin could be marked by introducing a small distortion and the time t could be signalled at x by running up a flag]

Rotating coordinates so that **x** lies along **k**

$$\phi = kx - \omega t$$

and

$$\left.\frac{dx}{dt}\right|_{\phi} = \frac{\omega}{k} = c \tag{1.5}$$

and is the phase velocity in the direction of **k**, normal to the wavefront. (Notice that the velocity with which the intersection of a plane of constant phase with any other axis travels is >c).

Then observer O (in the frame in which Maxwell's equations hold) measures a phase ϕ at x, t, and propagation vector **k**, frequency ω. An observer O' in some other frame measures the same value of the phase ϕ but $\mathbf{k}'$, ω' such that

$$\mathbf{k}'.\mathbf{x}' - \omega' t' \equiv \mathbf{k}.\mathbf{x} - \omega t$$

where the relation is an identity because the phase is the same number in all frames for any values of x, t. Using the Galilean relations $x' = x - vt$, $t' = t$ and equating coefficients we have

$$\mathbf{k}'.(\mathbf{x} - \mathbf{v}t) - \omega' t \equiv \mathbf{k}.\mathbf{x} - \omega t$$

and hence

$$\mathbf{k}' = \mathbf{k}, \quad \omega' = \omega - \mathbf{v}.\mathbf{k}' = \omega - \mathbf{v}.\mathbf{k} = \omega(1 - \frac{\mathbf{v}.\hat{\mathbf{k}}}{c}) \tag{1.6}$$

where $\hat{\mathbf{k}}$ is a unit vector normal to the wave front. This of course is the familiar Doppler shift. The phase velocity in S' is given by

$$c' = \frac{\omega'}{k'} = c - \mathbf{v}.\hat{\mathbf{k}} \qquad \text{(not equal to c)} \tag{1.7}$$

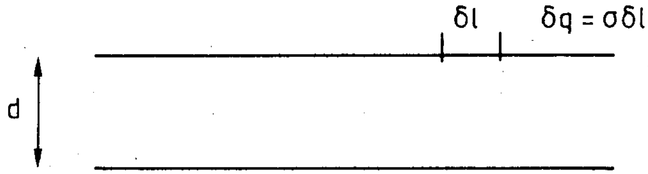
Note that c' is the phase velocity, normal to the wavefront and is not in general equal to the velocity of a pulse of light (or a photon) given by $\mathbf{u}' = \mathbf{c} - \mathbf{v}$ (although $\hat{\mathbf{k}}.\mathbf{u}' = c'$). In S' a pulse of radiation does not propagate normal to the wavefront (and it does if Maxwell's equations are true).

For propagation along the common x axis,

$$\sin(k'x' - \omega' t') = \sin(kx' - \omega(1 - \frac{v}{c})t')$$

and as $v \to c$ this equation predicts that an observer chasing the electromagnetic wave sees a field oscillating sinusoidally in space but not in time. This is not a solution of Maxwell's equations: the free field equations are not covariant with respect to the Galilean transformations.

Here is another example, in which sources of the fields are important. Consider the force per unit length acting between two (infinitely long) strings of charge, at rest in a frame where Maxwell's equations hold:



This is an elementary problem in electrostatics. The relevant Maxwell equation is

$$\nabla.\mathbf{E} = 4\pi\rho \qquad \text{where } \rho \text{ is (three dimensional) charge density}$$

$$\text{whence} \qquad \int \mathbf{E}.d\mathbf{S} = 4\pi q \qquad \text{where } q \text{ is the charge enclosed}$$

within the surface S. If the charge per unit length on a string is σ, then $\mathbf{E}$ is radial and given by

$$2\pi E(d) = 4\pi\sigma$$

$$E(d) = \frac{2\sigma}{d} \tag{1.8}$$

The force per unit length is

$$F_E = \frac{2\sigma^2}{d} \qquad \textit{outward} \tag{1.9}$$

Now suppose the strings move in the direction of their lengths with velocity v relative to a frame in which Maxwell's equations hold. The electrical force still takes the form (1.9), but there is also a magnetic force between the two strings. For steady currents

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}$$

and applying Stokes' theorem

$$2\pi d B(d) = \frac{4\pi}{c} I$$

$$B(d) = \frac{2I}{cd} \tag{1.10}$$

The force is $IB(d)/c$ per unit length, or

$$F_B = \frac{2I^2}{c^2 d} \qquad \textit{inward} \tag{1.11}$$

and I (charge per second) is given by

$$I = \sigma v$$

so that the total force per unit length is

$$F = \frac{2\sigma^2}{d}\left(1 - \frac{v^2}{c^2}\right) \tag{1.12}$$

[The two infinitely long strings were chosen to get rid of retardation effects. The results are correct in a frame in which Maxwell's equations are true, although the result for the electric field of a moving string may surprise you if you know the result for the transverse field of a moving (point like) particle]. If the measured value of σ is the same in the Maxwell frame and in a frame moving with the strings, then the two different values obtained for F are inconsistent with Galilean relativity and again the appropriate Maxwell equations could not hold in the moving frame.

Since the validity of a universal principle of relativity is to be determined experimentally, consider the results of experiments which tested explicitly for the existence of a preferred frame. The original experiment was that of Michelson and Morley, in which a Michelson interferometer with two equal length arms at right angles was rotated through 90° while the fringes formed were observed. A movement of the fringes by an amount proportional to $\frac{v^2}{c^2}$, where v is the velocity of the interferometer through the aether (or with respect to the Maxwell frame) was expected: no fringe shift was observed.

[PROBLEM: Work out the expression for the fringe shift as a function of the arm lengths, wavelength of light employed and velocity v. Evaluate it for arm lengths of 1m and $v \sim 30\,\mathrm{km\,s^{-1}}$ (orbital velocity of the earth), $v \sim 200\,\mathrm{km\,s^{-1}}$ (velocity of the sun around the galaxy) and $v \sim 400\,\mathrm{km\,s^{-1}}$ (velocity of the sun with respect to the microwave background).]

The most sensitive experiment looking for motion relative to a preferred frame employed the Mössbauer effect to look for Doppler shift variations, eq. (1.6). 14.4 KeV photons from the decay of ^{57}Fe (fed by β emission from ^{57}Co) have a natural line width $\frac{\Delta\omega}{\omega} \sim 10^{-13}$ and at room temperature the iron nuclei are sufficiently locked into the crystal lattice that the recoil momentum is taken up by a patch of crystal rather than by the daughter nucleus. First order Doppler shifts due to thermal jiggling are also absent. If an absorbing iron foil (enriched in ^{57}Fe) is moved towards or away from a 14.4 KeV ^{57}Fe source, the change in absorbtion as a function of velocity is sensitive to velocities $\ll 0.1\,\mathrm{mm\,s^{-1}}$.

Consider a source moving through the preferred frame with velocity $\mathbf{v}_s$, frequency (in the source frame) ω_s. Then from (1.6)

$$\omega_s = \omega\left(1 - \frac{\mathbf{v}_s}{c}.\hat{\mathbf{k}}\right)$$

where ω is the frequency in the Maxwell frame. The frequency measured in the absorber frame is similarly

$$\omega_a = \omega\left(1 - \frac{\mathbf{v}_a}{c}.\hat{\mathbf{k}}\right)$$

so that

$$\omega_s - \omega_a = \omega\frac{\mathbf{v}_a - \mathbf{v}_s}{c}.\hat{\mathbf{k}}$$

$\mathbf{v}_s = \mathbf{u}_s - \mathbf{v}$, $\mathbf{v}_a = \mathbf{u}_a - \mathbf{v}$ where $\mathbf{u}_s$, $\mathbf{u}_a$ are the laboratory velocities of the source and absorber and $\mathbf{v}$ is the velocity of the laboratory through the preferred frame. Thus

$$\omega_s - \omega_a = \omega\left(\frac{\mathbf{u}_a - \mathbf{u}_s}{c}\right).\hat{\mathbf{k}} \qquad (1.13)$$

At first sight, (1.13) is independent of $\mathbf{v}$. This is not in fact the case, for the scalar product contains dependence on $\mathbf{v}$. The direction of travel of the photons $\hat{\mathbf{p}}$ is offset from $\hat{\mathbf{k}}$ by $\sim \frac{\mathbf{v}}{c}$ and so if $\hat{\mathbf{p}}.(\mathbf{u}_a - \mathbf{u}_s) = 0$ then

$$\omega_s - \omega_a = \omega(\mathbf{u}_a - \mathbf{u}_s).\frac{\mathbf{v}}{c^2} \qquad (1.14)$$

In the aether drift experiment, the source and absorber were mounted at opposite ends of the diameter of a turntable which could be rotated at high speed. Then $\hat{\mathbf{p}}.(\mathbf{u}_a - \mathbf{u}_s)$ is indeed zero and so $\frac{\omega_a - \omega_s}{\omega}$ will oscillate with the rotational frequency of the turntable and an amplitude proportional to the component of $\mathbf{v}$ in the plane of the turntable. A limit $v < 5\,\mathrm{cm\,s^{-1}}$ has been set [Reference G. R. Isaak, *Phys. Bull.* **21** 255 (1970)].

[PROBLEM: (There is no easy way to solve this—it has to be worked out carefully and requires some playing around. These sort of considerations are of enormous importance in real experimental physics). At first sight, $\hat{\mathbf{p}}.(\mathbf{u}_a - \mathbf{u}_s) = 0$ seems obvious for the turntable — the photon travels along a diameter, doesn't it? The answer is no, because of the finite speed of light in the laboratory, which changes with orientation of the source and absorber relative to $\mathbf{v}$. So first show that $\hat{\mathbf{p}}.(\mathbf{u}_a - \mathbf{u}_s)$ really is zero, for source and absorber opposite, each at a radius R, independent of the velocity with which the photon travels in the laboratory. But you will never locate them precisely at a distance R, so take radii R_s and R_a and show that $\hat{\mathbf{p}}.(\mathbf{u}_a - \mathbf{u}_s)$ is still zero. Then remember that the source and absorber will not be precisely located on a diameter (and in any case cover finite areas) so show that even so $\hat{\mathbf{p}}.(\mathbf{u}_a - \mathbf{u}_s)$ is zero!]

So experiments designed to detect the motion of a terrestrial laboratory relative to an aether frame reveal no such effect — what do we do? Make some guesses (and in the early stages we don't worry if the philosophers tell us our hypotheses are *ad hoc*.)

1. Perhaps an aether (with mechanical properties) is dragged along as a boundary layer with the earth? NO (because the apparent angular position of stars changes with a period of 1 year — stellar parallax.)

2. Perhaps the preferred frame is always the emitter frame? NO (initially ruled out by observation of binary stars, but this evidence is probably vitiated by the phenomenon of extinction — light is absorbed and reradiated by the interstellar medium and the memory of source velocity would disappear in a few light years. But the velocity of photons from the decay of $\pi^{0\prime}s$ travelling with a velocity $\sim c$ in the laboratory has been measured and is $\sim c$ rather than $2c$ [T. Alvager et al., *Phys. Lett.* **12** 260 (1964)])

3. Given a theory with a preferred frame, do we really expect to MEASURE a direction dependent velocity?

Lorentz's answer to this question was NO, because Maxwell's equations, taken as true in one frame, lead us to expect a longitudinal contraction of apparatus moving with respect to this frame (and so the Lorentz contraction is more than an *ad hoc* hypothesis).

Suppose a measuring stick (or the arm of a Michelson interferometer) consists of charges sitting at the bottom of electrostatic potential wells. In regions of space where the charge density is zero, the scalar potential may be chosen to be a solution of the equation

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{1.15}$$

in the Maxwell frame.

The scalar potential in a measuring stick at rest with respect to the aether thus satisfies the Laplace equation

$$\nabla^2\phi = 0$$

In motion, the scalar potential ϕ_M of the stick must satisfy (1.15) in the Maxwell frame, because the potential at fixed $\mathbf{x}$ changes with time. If the stick moves in the $\mathbf{x}$ direction with velocity v, then

$$\phi_M = \phi_M(x \pm vt)$$

and so

$$\frac{\partial^2 \phi_M}{\partial t^2} = v^2 \frac{\partial^2 \phi_M}{\partial x^2}$$

Freezing the picture at a given instant, the potential must satisfy the equation

$$\frac{\partial^2 \phi_M}{\partial x^2}\left(1 - \frac{v^2}{c^2}\right) + \frac{\partial^2 \phi_M}{\partial y^2} + \frac{\partial^2 \phi_M}{\partial z^2} = 0 \tag{1.16}$$

this pattern moving with velocity v through the aether. This equation is just Laplace's equation again, in scaled coordinates

$$\nabla_M^2 \phi_M = 0$$

where

$$x_M = \frac{x}{\sqrt{1 - v^2/c^2}}, \quad y_M = y, \quad z_M = z.$$

We may therefore set $\phi_M(\mathbf{x}_M) = \text{const}\ \phi(\mathbf{x})$ and the distance between maxima and minima of ϕ_M is the same, measured in terms of x_M, as the separation between maxima and minima of ϕ, measured in terms of the true Maxwell frame coordinate x. Thus for the moving rod, the distance between maxima and minima measured in terms of x is reduced by a factor $\sqrt{1 - v^2/c^2}$. This is precisely what is needed to account for the null result of the Michelson-Morley experiment with equal arm lengths. This result suggests that charge density is likely to be increased (in order to match the equipotentials) and if for a moving string of charge $\sigma_M = \frac{\sigma}{\sqrt{1-v^2/c^2}}$ THEN we no longer have any problem with the (transverse) force per unit length acting between strings of charge (eq. (1.12)). [This also removes the apparent contradiction between the transverse fields of a moving point particle and eq. (1.8)].

This is of course not the whole story. A purely static distribution of charges is not stable, but there are a number of other interesting effects which are relevant. The momentum in the electromagnetic field of a point (or very small) particle can be calculated via Maxwell's equations and it rises faster than linearly with velocity in the Maxwell frame, giving rise to different transverse and longitudinal masses. The shape of the orbit of a bound state of two charges moving through the Maxwell frame can be calculated and indeed is shrunk by a factor $\sqrt{1 - v^2/c^2}$ (where v is the velocity of the centre of mass of the pair through the aether) AND the period is increased, relative to an identical molecule at rest. Setting aside the problems of constructing a consistent classical theory of electromagnetic matter, there is every reason to believe that if Maxwell's equations are true in a given frame, then pure electromagnetic measuring sticks are shrunk when moving relative to this frame and pure electromagnetic clocks are slowed. Electromagnetism is so constructed that pure electromagnetic matter does not allow you to detect a preferential frame at all, and provided measurements are made with electromagnetically constructed instruments, Maxwell's equations will be equally valid in all inertial frames, but coordinates so measured will not be related by the Galilean transformations.

Natural reactions:

(1) Horribly complicated.

It is indeed, but this is the way the Lorentz transformations were first obtained. It is fascinating to read the accounts of these heroic struggles given by Lorentz and by Larmor — despite the (charmingly) archaic phraseology, they speak to the heart of anyone who has lived through the last twenty years of particle physics.

(2) But matter is not pure electromagnetic anyway.

True, but perhaps nature has organised a conspiracy so that we can never detect these effects with anything? Then we could forget about preferred frames. We make measurements with real rods and clocks and in motion they do funny things relative to Galilean space. Maxwell's equations (interpreted in terms of real measurements with apparatus constructed from electromagnetism) are covariant under a set of transformations other than the Galilean — the Lorentz transformations. If there is a universal conspiracy, we may adopt the point of view that velocity does the same funny things to all clocks and rods or we may work directly in terms of real measurements and relate the coordinates of an event measured in two different inertial frames through the Lorentz transformations rather than the Galilean. These two points of view are operationally indistinguishable. We shall investigate the second — special relativity. We shall find the Lorentz transformations of course, and a corollary. If a principle of relativity embodying the Lorentz transformations holds universally, then Newtonian mechanics requires modification.

GENERAL REFERENCES

Lorentz's viewpoint may be found in a paper (1904) reprinted in

The Principle of Relativity, A. Einstein et al (Dover 1952) — still in print.

It is also discussed in

The Theory of Electrons, H. A. Lorentz (2nd ed. Leipzig, New York 1916)

Larmor's work is contained in

Aether and Matter, J. Larmor (Cambridge 1900)

These two books are relatively rare but can be found in some libraries. A modern account of this approach is given by

J. S. Bell, *Prog. Scientific Culture* **1/2** 135 (1976)

Lecture 2 THE LORENTZ TRANSFORMATIONS

In the first lecture we more or less defined the concept of a principle of relativity, defined inertial frames, investigated the Galilean transformations and Newton's laws and then passed on to electromagnetism. We found that even in an aether theory there are good grounds for expecting the null result of the Michelson-Morley experiment because the theory predicts the contraction of measuring rods (and the slowing of clocks) travelling through the aether. The contraction also fixes up the force between moving strings of charge and in no case is motion with respect to the aether observable. If all forms of matter are identically affected by velocity, then we may retain a principle of relativity.

We retain the idea of inertial frames, in which particles which are not acted on by any force move with constant velocity and in which physics does not depend on either the origin or orientation of the coordinate system. We postulate that Maxwell's equations take the same form, with the same numerical content, in all such inertial frames, and seek a relation between measured coordinates (x_i, t) in one inertial frame S and the measured coordinates of the same event (x'_i, t') in another frame S', such that this condition is satisfied; in particular the measured velocity of light is a universal constant. In special relativity we further postulate that all inertial frames are equivalent, that the whole of physics is governed by equations equally true for all unaccelerated observers and that these equations retain their form and numerical content under the transformations we are seeking. Because we have learned that in an aether theory funny things happen to rods and clocks as a function of velocity, the transformations we seek need not accord with our intuition (but they had better reduce to the Galilean transformations in the low velocity limit).

Philosophically, we are shifting our view of space and time from one in which they are marked off in absolute units, to which a key exists, to an operational view in which space is that which is measured with measuring sticks and time is that which is measured with clocks. It is convenient and appropriate, seems obvious once spelt out, but is not strictly necessary.

(x_i, t) are the coordinates of an event measured by a given observer and his assistants in an inertial frame S, using standard instruments. (For example, the unit of length could be 10^8 ^{12}C atoms, the unit of time a reciprocal atomic frequency, and clocks at different places in S synchronised using the laws of physics obtaining in S). The same event (make it a supernova explosion if you want to be spectacular) has coordinates (x'_i, t') measured by an observer in S', using instruments constructed according to the same physical specification as those in S. What must the relation be between (x_i, t) and (x'_i, t') if both observers find Maxwell's equations are true?

There is no correct way of deriving these transformations (the Lorentz transformations). Lorentz discovered them by playing about with Maxwell's equations, a long and hard route. The important thing is the output, not the input. It is perfectly in order to make plausible guesses, so long as the answer is satisfactory. A rigorous derivation can only proceed from a set of assumptions, usually with the benefit of hindsight, and here we shall work from two very simple assumptions. The first (abstracted from Maxwell's equations) is that the speed of light is a universal constant c, measured in any inertial frame. The second assumption is that the transformations are linear in (x_i, t), (x'_i, t').

A general transformation is

$$x'_i = f_i(x_j, t)$$
$$t' = f_4(x_j, t)$$