# CONDUCTION
# OF
# HEAT IN SOLIDS

PRINTED IN GREAT BRITAIN
AT THE UNIVERSITY PRESS, OXFORD
BY VIVIAN RIDLER
PRINTER TO THE UNIVERSITY

# PREFACE TO THE SECOND EDITION

THE death of Professor Carslaw in 1954 has left me with the task of preparing a new edition of this book. In doing this, I have attempted, while preserving so far as possible the form and spirit of Carslaw's mathematics, to provide as complete an account as possible of the exact solutions and soluble problems of the subject. To this end, a great many new results have been added and some parts of the discussion have been greatly expanded, for example, those on heat generation, surface heating, melting and freezing, geophysical applications, anisotropic media, moving media, and substances with variable thermal properties.

A number of new tables and text-figures giving numerical information on fundamental problems has been added. The number of references has now grown to over seven hundred; it is quite impossible to refer to all works on the subject, and the papers listed are largely confined to those which I have been able to consult, but I have attempted to give an adequate coverage of all branches of the subject.

Two short survey chapters have been added. The first of these gives an introduction to the integral transform notation and its relationship to the classical Fourier methods. The second gives an account of the numerical methods which have assumed great importance in the last decade and their relationship to the body of exact solutions given earlier in the text.

It is a pleasure to acknowledge the assistance of my wife and Mrs. A. Davidson in the preparation of the manuscript and the numerical calculations, and that of the staff of the Clarendon Press in the production of the book.

J. C. J.

# PREFACE TO THE FIRST EDITION

CARSLAW'S *Introduction to the Theory of Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat* was published at the end of 1906. In 1920 and 1921 the work was completely revised and rewritten in two volumes, the second of which, entitled *Introduction to the Mathematical Theory of the Conduction of Heat in Solids*, appeared in 1921. It became out of print in 1940.

In the last twenty-five years so many developments have been made, both in the theory and applications of the subject, that a new book embodying these advances seemed called for rather than a new and revised edition of the old one. This work, based on the earlier one and intended to supersede it, brings the discussion of the theory and applications up to date. In particular it contains a full treatment of the Laplace transformation method of dealing with problems in the Conduction of Heat. This takes the place of the method by contour integrals given in Chapters X and XI of the 1921 book. The Laplace transformation method, though similar in principle to that by contour integration, is much simpler, more direct and powerful.

In planning this book we have tried to make it as useful as possible to engineers and physicists without altering its character as a mathematical work. Explicit solutions of many problems of practical interest are included and much numerical information in the form of tables and text-figures is given. The discussion of the theory of systems used in experimental work has been greatly extended, and other subjects of practical importance are briefly noticed, such as the theory of automatic temperature control, which have hitherto not appeared in mathematical textbooks.

The earlier book, except in its final chapters, could be looked on as a treatise on the Fourier mathematics, developing the subject along the classical lines. The new book in Chapters I–X follows the same design. In these chapters it covers and often reproduces verbatim most of what is contained in Chapters I–IX of the old one, while giving fuller attention to the needs of the engincer and physicist.

In Chapters XII–XV the Laplace transformation method is introduced and applied in the main to more difficult problems. The reader, after the general discussion in Chapter XII, will see that its use would have simplified much of the preceding chapters, and he will probably, in the solution of problems as they arise, become accustomed to use it for himself.

A large number of interesting results has been given in small print, many without proof. These may be taken as examples for solution.

Over four hundred selected references to the mathematical and physical aspects of the various topics discussed here have been given as footnotes in the text. It is hoped that these will prove an adequate introduction to the literature of the subject. This has grown so much in recent years that it seemed impossible to give a complete bibliography.

Almost all the numerical material in the tables and text-figures has been calculated specially for this book. We are greatly indebted to Miss M. E. Clarke for her assistance in this computation and in many other ways.

H. S. C.
J. C. J.

# CONTENTS

# I

# GENERAL THEORY

## 1.1. Introductory

WHEN different parts of a body are at different temperatures heat flows from the hotter parts to the cooler. There are three distinct methods by which this transference of heat takes place: (i) Conduction, in which the heat passes through the substance of the body itself, (ii) Convection, in which heat is transferred by relative motion of portions of the heated body, and (iii) Radiation, in which heat is transferred direct between distant portions of the body by electromagnetic radiation.

In liquids and gases convection and radiation are of paramount importance, but in solids convection is altogether absent and radiation usually negligible. In this book we shall consider conduction of heat only, and usually speak of the body as solid, though in certain circumstances the results will be valid for liquids or gases.

In this chapter the general theory of conduction of heat is developed; the subsequent chapters are devoted to special problems and methods.

## 1.2. Conductivity

The Mathematical Theory of the Conduction of Heat may be said to be founded upon a hypothesis suggested by the following experiment:

A plate of some solid is given, bounded by two parallel planes of such an extent that, so far as points well in the centre of the planes are concerned, these bounding surfaces may be supposed infinite. The two planes are kept at different temperatures, the difference not being so great as to cause any sensible change in the properties of the solid. For example, the upper surface may be kept at the temperature of melting ice by a supply of pounded ice packed upon it, and the lower at a fixed temperature by having a stream of warm water continually flowing over it. When these conditions have endured for a sufficient time the temperature of the different points of the solid settles down towards its steady value, and at points well removed from the ends the temperature will remain the same along planes parallel to the surfaces of the plate.

Consider the part of the solid bounded by an imaginary cylinder of cross-section $S$ whose axis is normal to the surface of the plate. This cylinder is supposed so far in the centre of the plate that no flow of heat takes place across its generating lines. Let the temperature of the lower surface be $v_0$° C and of the upper $v_1$° C ($v_0 > v_1$), and let the thickness of the plate be $d$ centimetres. The results of experiments upon different solids suggest that, when the steady state of temperature has been reached, the quantity $Q$ of heat which flows up through the plate in $t$ seconds over the surface $S$ is equal to

$$\frac{K(v_0-v_1)St}{d}, \tag{1}$$

where $K$ is a constant, called the Thermal Conductivity of the substance, depending upon the material of which it is made. In other words, the flow of heat between these two surfaces is proportional to the difference of temperature of the surfaces.

This result must not be regarded as proved by these experiments. They suggest the law rather than verify it. The more exact verification is to be found in the agreement of experiment with calculations obtained from the mathematical theory based on the assumption of the truth of this law.

The reciprocal of the thermal conductivity of a substance is called its Thermal Resistivity.

Strictly speaking, the conductivity $K$ is not constant for the same substance, but depends upon the temperature. However, when the range of temperature is limited, this change in $K$ may be neglected, and in the ordinary mathematical theory it is assumed that the conductivity does not vary with the temperature. A nearer approximation to the actual state may be obtained by making $K$ a linear function of the temperature $v$, e.g.

$$K = K_0(1+\beta v),$$

where $\beta$ is small, and, in fact, is negative for most substances.

From (1), the thermal conductivity is given by

$$K = \frac{Qd}{(v_0-v_1)St}, \tag{2}$$

and from this its dimensions and the nature of the units in which it is expressed follow.

The system of units most frequently chosen in physical work uses the c.g.s. units of length, mass, and time, measures temperature in ° C, and takes as the unit quantity of heat the calorie, which is the quantity of heat required to raise the temperature of 1 gm of water† by 1° C. In this system, values of $K$ are expressed in cal/(sec) (cm$^2$) (° C/cm). This system will be used throughout this book when numerical values are given: values of the thermal properties of a few typical substances‡ are given in Appendix VI to give an idea of the orders of magnitude involved.

The other important system of units, which is the one commonly employed by engineers and used in works on heat transfer, takes the foot, pound, hour, and ° F as units, and defines the unit quantity of heat as the British Thermal Unit (Btu.), which is the quantity of heat required to raise the temperature of 1 lb of water at its maximum density (39° F) by 1° F. The connexion between the two units is

$$1 \text{ Btu.} = 252{\cdot}0 \text{ cal.}$$

In this system, numerical values of the conductivity are given in Btu./(hr)(ft.$^2$)(° F/ft.): to express these in terms of cal/(sec) (cm$^2$)(° C/cm) multiply by 0·00413.

The dimensions of $K$ in these systems in which the unit of heat is that which causes unit rise in temperature in unit mass of water may be seen from (2) to be

$$[K] = [M][L^{-1}][T^{-1}], \tag{3}$$

since those of $Q/(v_0 - v_1)$ are just those of mass.

If it is desired to measure quantity of heat by the work necessary to produce it, the unit would be the erg or joule. The number of joules in a calorie, $J$, is known as the mechanical equivalent of heat. For the 15° calorie defined above $J = 4{\cdot}184$.

In the fundamental experiment from which our definition of the conductivity is derived, the solid is supposed to be homogeneous and of such a material that, when a point within it is heated, the heat spreads out equally well in all directions. Such a solid is said to be isotropic, as opposed to crystalline and anisotropic solids, in which certain directions are more favourable for the conduction of heat than others. There are also heterogeneous solids, in which the conditions of conduction vary from point to point as well as in direction at each point.

† Experiments show that the quantity of heat required to raise the temperature of 1 gm of water by 1° C is not quite the same at different temperatures, and in an exact definition of the calorie the temperature of the water needs to be specified. Usually, this specified temperature is taken to be 15° C so that the '15° calorie' is the quantity of heat needed to raise 1 gm of water from 14·5° C to 15·5° C.

‡ For more extensive information see the *International Critical Tables* (McGraw-Hill, 1929), Vol. V, or, for rocks and minerals, Birch, Schairer, and Spicer, *Handbook of Physical Constants*, Geol. Soc. of America, Special papers, Number 36 (1942).

## 1.3. The flux of heat across any surface

*The rate at which heat is transferred across any surface S at a point P, per unit area per unit time, is called the flux of heat*† *at that point across that surface, and we shall denote it by f.*

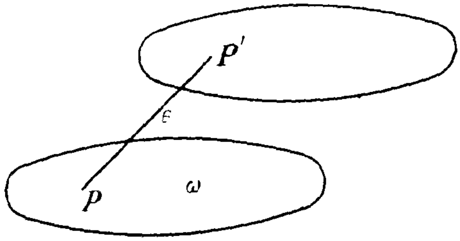


FIG. 1

First we show that the flux across a plane through a point $P$ varies continuously with the position of the point $P$ if the direction of the normal to the plane remains constant. Suppose an infinitesimal area $\omega$ enclosing $P$ is taken in the plane, and a cylinder is formed on this area as base with generators equal and parallel to a line $PP'$ whose length $\epsilon$ is an infinitesimal of lower order than the linear dimensions of $\omega$ (Fig. 1).

Let $f_1 \omega$ and $f_2 \omega$ be the rates of flow per unit time across the plane surfaces of the cylinder through $P$ and $P'$. The flow across the curved surface is negligible compared with these. The rate of flow of heat into the cylinder is thus $\omega(f_1-f_2)$. Also if $v$ is the average temperature in the cylinder, $\sigma$ the distance between its plane faces, and $\rho$ and $c$ the average density and specific heat of its material, the rate at which the cylinder gains heat is

$$\rho c \omega \sigma \frac{\partial v}{\partial t}.$$

Equating these two expressions we have

$$f_1 - f_2 = \rho c \sigma \frac{\partial v}{\partial t},$$

and as $\sigma \to 0$ the expression on the right tends to zero and so $f_1 \to f_2$.

It is important to notice that this argument does not require the thermal properties of the medium to be continuous, only that they be finite. Thus it enables us to assert in § 1.9 that the flux is continuous at the surface of separation of two media.

Next we show that, if the values of $f$ are given for three mutually

† Numerical values are usually given in cal/(cm²)(sec) or in Btu./(ft.²)(hr). The connexion between the two is 1 cal/(cm²)(sec) = 13,270 Btu./(ft.²)(hr).

perpendicular planes meeting at a point, its value for any other plane through the point may be written down.

Consider the elementary tetrahedron $PABC$, Fig. 2, whose three faces $PBC$, $PCA$, $PAB$ are parallel to the coordinate planes, while the perpendicular from the point $P$ to the face $ABC$ has direction cosines $(\lambda, \mu, \nu)$ and is of length $p$. Let the area of $ABC$ be $\Delta$; then the areas of $PBC$, $PCA$, and $PAB$ are respectively $\lambda\Delta$, $\mu\Delta$, $\nu\Delta$.

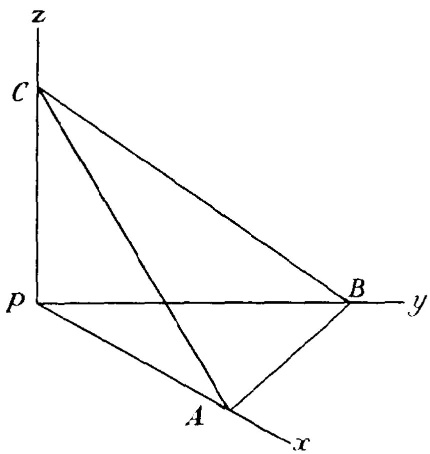


FIG. 2

If we denote the rates of flow of heat per unit time per unit area over the elementary areas $PBC$, $PCA$, $PAB$, and $ABC$ by $f_x$, $f_y$, $f_z$, and $f$, the rate at which heat is gained by the tetrahedron is given by

$$(\lambda f_x+\mu f_y+\nu f_z-f)\Delta.$$

However, if $\rho$ and $c$ are the density and specific heat of the solid, and $v$ the average temperature over the tetrahedron, this rate of gain of heat is equal to

$$\tfrac{1}{3}\Delta p\rho c\frac{\partial v}{\partial t}.$$

It follows that
$$\lambda f_x+\mu f_y+\nu f_z-f = \tfrac{1}{3}p\rho c\frac{\partial v}{\partial t}. \tag{1}$$

Now as $p$ tends to zero, the right-hand side of (1) tends to zero, and $f_x$, $f_y$, $f_z$, and $f$ become respectively the fluxes at the point $P$ across planes parallel to the coordinate planes and across a plane through $P$ which has $\lambda$, $\mu$, $\nu$ for the direction cosines of its normal. Thus we have

$$f = \lambda f_x+\mu f_y+\nu f_z. \tag{2}$$

If the three fluxes $f_x, f_y, f_z$ at a point $P$ across planes parallel to the coordinate planes are known, the flux across any other plane through $P$ can be determined from (2).

At every point $P$ of the solid a vector **f** is defined whose components are $f_x, f_y, f_z$. Its magnitude is

$$f_m = \sqrt{(f_x^2+f_y^2+f_z^2)}, \tag{3}$$

and it lies along the line whose direction cosines are

$$f_x/f_m, \quad f_y/f_m, \quad f_z/f_m. \tag{4}$$

This vector may be called the *flux vector* at the point $P$. The flux of heat at $P$ across a plane whose normal lies in the direction (4) is just $f_m$, while the flux at $P$ across a plane whose normal makes an angle $\theta$ with this direction is $f_m \cos\theta$.

## 1.4. Isothermal surfaces

Consider a solid with a distribution of temperature at time $t$ given by

$$v = f(x, y, z, t).$$

We may suppose a surface described in this solid, such that at every point upon it the temperature at this instant is the same, say $V$. Such a surface is called the isothermal surface for the temperature $V$, and it may be looked upon as separating the parts of the body which are hotter than $V$ from the parts which are cooler than $V$. We may imagine the isothermals drawn for this instant for different degrees and fractions of a degree. These surfaces may be formed in any way, but no two isothermals can cut each other, since no part of the body can have two temperatures at the same time. The solid is thus pictured as divided up into thin shells by its isothermals.

## 1.5. Conduction of heat in an isotropic solid

In future, unless expressly stated, we shall consider only isotropic media, that is media whose structure and properties in the neighbourhood of any point are the same relative to all directions through the point. Because of this symmetry, the flux vector at a point must be along the normal to the isothermal surface through the point, and in the direction of falling temperature.

The relation between the rate of change of temperature along the normal to an isothermal and the flux vector in that direction may be deduced from the fundamental experiment described in § 1.2. In that case the isothermals are planes parallel to the faces of the slab. Suppose the isothermals for temperatures $v$ and $v+\delta v$ are distant $\delta x$ apart.

Then by 1.2 (1) the rate of flow of heat, per unit time per unit area, in the direction of $x$ increasing is

$$-K\frac{\delta v}{\delta x}.$$

Thus in the limit as $\delta x \to 0$ we have

$$f_x = -K\frac{\partial v}{\partial x}. \tag{1}$$

We extend this to any isothermal surface and take as our *fundamental hypothesis for the Mathematical Theory of Conduction of Heat that the rate at which heat crosses from the inside to the outside of an isothermal surface per unit area per unit time at a point is equal to*

$$-K\frac{\partial v}{\partial n},$$

*where $K$ is the thermal conductivity of the substance, and $\partial/\partial n$ denotes differentiation along the outward-drawn normal to the surface.*

We now proceed to find the flux at a point $P$ across any surface, not necessarily isothermal. Let the tangent plane at $P$ to the isothermal through $P$ be taken as the $XY$-plane, so that the fluxes across planes through $P$ parallel to the coordinate planes are

$$f_x = f_y = 0, \qquad f_z = -K\frac{\partial v}{\partial z}.$$

Then if the normal at $P$ to the given surface has direction cosines $(\lambda, \mu, \nu)$ relative to these axes, the flux across it is, by 1.3 (2),

$$-K\nu\frac{\partial v}{\partial z} = -K\frac{\partial v}{\partial h},$$

where $\partial/\partial h$ denotes differentiation in the direction $(\lambda, \mu, \nu)$, since

$$\frac{\partial v}{\partial h} = \lambda\frac{\partial v}{\partial x}+\mu\frac{\partial v}{\partial y}+\nu\frac{\partial v}{\partial z}, \quad \text{and} \quad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

Thus *the flux of heat at a point across any surface is*

$$-K\frac{\partial v}{\partial h}, \tag{2}$$

*where $\partial/\partial h$ denotes differentiation in the direction of the outward normal.*

In particular, the fluxes across three planes parallel to the axes of coordinates are

$$f_x = -K\frac{\partial v}{\partial x}, \qquad f_y = -K\frac{\partial v}{\partial y}, \qquad f_z = -K\frac{\partial v}{\partial z}. \tag{3}$$

Using the vector **f** introduced in § 1.3, the results of this section may be expressed by the formula

$$\mathbf{f} = -K \operatorname{grad} v. \tag{4}$$

## 1.6. The differential equation of conduction of heat in an isotropic solid

We first consider the case of a solid through which heat is flowing, but within which no heat is generated. The temperature $v$ at the point $P\,(x,y,z)$ will be a continuous function of $x$, $y$, $z$, and $t$, and, as shown in § 1.3, the same is true of the flux.

Consider an element of volume of the solid at the point $P$, namely, the rectangular parallelepiped with this point as centre, its edges being parallel to the coordinate axes and of lengths $2dx$, $2dy$, and $2dz$. Let $ABCD$ and $A'B'C'D'$ be the faces in the planes $x-dx$ and $x+dx$, respectively, then the rate at which heat flows into the parallelepiped over the face $ABCD$ will ultimately be given by

$$4\left(f_x - \frac{\partial f_x}{\partial x}\,dx\right) dydz,$$

where $f_x$ is the flux at $P$ over a parallel plane. Similarly, the rate at which heat flows out over the face $A'B'C'D'$ is given by

$$4\left(f_x + \frac{\partial f_x}{\partial x}\,dx\right) dydz.$$

Thus the rate of gain of heat from flow across these two faces is equal to

$$-8\,dxdydz\,\frac{\partial f_x}{\partial x}.$$

There are similar expressions for the rates of gain from flow across the other pairs of faces, and, adding these, the total rate of gain of heat of the parallelepiped from flow across its faces is found to be

$$-8\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right) dxdydz = -8\,dxdydz\,\operatorname{div}\mathbf{f}, \tag{1}$$

where **f** is the vector defined in 1.3 (4).

This rate of gain of heat is also given by

$$8\rho c\,\frac{\partial v}{\partial t}\,dxdydz, \tag{2}$$

where $\rho$ is the density and $c$ the specific heat† (at temperature $v$) of the solid. Equating (1) and (2) gives‡

$$\rho c \frac{\partial v}{\partial t} + \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right) = 0. \tag{3}$$

This equation holds at any point of the solid, provided no heat is supplied at the point; the solid need not be homogeneous or isotropic. It corresponds to the equation of continuity in hydrodynamics.

For a homogeneous isotropic solid whose thermal conductivity is independent of the temperature, $f_x$, $f_y$, and $f_z$ are given by 1.5 (3), and (3) becomes

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} - \frac{1}{\kappa}\frac{\partial v}{\partial t} = 0, \tag{4}$$

where

$$\kappa = \frac{K}{\rho c}. \tag{5}$$

The constant $\kappa$ was called by Kelvin the Diffusivity§ of the substance, and by Clerk Maxwell its Thermometric Conductivity.‖

(4) is the equation commonly known as the equation of conduction of heat. In the case of steady temperature in which $v$ does not vary with the time, it becomes Laplace's equation

$$\nabla^2 v \equiv \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0. \tag{6}$$

† The specific heat $c$ of a substance at temperature $v$ is defined as $\delta Q/\delta v$ where $\delta Q$ is the quantity of heat necessary to raise the temperature of unit mass of the substance through the small temperature range from $v$ to $v+\delta v$. It depends on both the temperature and the assumed mode of heating, which is taken here to be at constant strain. In the units employed in this book it is expressed in (cal)/(gm)(°C), and the specific heat of water at 15° C is 1 (cal)/(gm)(° C). It should be noticed that there is a considerable variety of usage in this matter. Some writers regard the above definition as that of heat capacity, or heat capacity per unit mass, of the substance, and define the specific heat of a substance as the ratio of its heat capacity per unit mass to that of water.

For solids, the effect of the method of heating on the specific heat is usually negligible and $c$ may be replaced by $c_p$, the specific heat at constant pressure. The question is discussed in III at the end of this section.

‡ In transparent, as well as in fibrous or other materials with large pore spaces, transport of heat by radiation may be of importance, resulting in the appearance of an additional term in (3). Cf. van der Held, *Appl. Sci. Res.* A, **3** (1953) 237–49; A, **4** (1954) 77–99.

§ Some values for the diffusivities of various substances are given in Appendix VI. To find the dimensions of diffusivity we note that, writing $[Q]$ and $[v]$ for those of quantity of heat and temperature, respectively, $[K] = [Q][L^{-1}][T^{-1}][v^{-1}]$, $[c] = [Q][M^{-1}][v^{-1}]$, $[\rho] = [M][L^{-3}]$, so that $[\kappa] = [L^2][T^{-1}]$. It follows that if the units of length and time are the foot and the hour, as in many engineering tables, the value of $\kappa$ for these units will have to be multiplied by $(30{\cdot}48)^2/3600 = 0{\cdot}258$ to reduce it to the c.g.s. system.

‖ Because it measures the change of temperature which would be produced in unit volume of the substance by the quantity of heat which flows in unit time through unit area of a layer of the substance of unit thickness with unit difference of temperature between its faces.

If heat is produced in the solid, so that at the point $P$ $(x, y, z)$ heat is supplied at the rate $A(x, y, z, t)$ per unit time per unit volume, a term $8A\, dxdydz$ has to be added to (1), and, for the case in which $K$ is a constant, (4) is replaced by

$$\nabla^2 v - \frac{1}{\kappa}\frac{\partial v}{\partial t} = -\frac{A(x, y, z, t)}{K}. \tag{7}$$

For the case of steady flow, in which $\partial v/\partial t = 0$, equation (7) reduces to Poisson's equation.

In almost all the problems for which exact solution is possible, and in those discussed in this book unless otherwise stated, the thermal properties $K$, $\rho$, $c$ are constants, independent of both position and temperature. If this is not the case, (3) still holds (with $A(x, y, z, t)$ added in the right-hand side if there is heat generation) but (7) is replaced by

$$\rho c \frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(K \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(K \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(K \frac{\partial v}{\partial z}\right) + A. \tag{8}$$

If $K$ and $A$ are functions of position only, the solution of (8) offers no great difficulty in principle and a number of solutions are available for discontinuous thermal properties (composite solids) and for simple laws of variation of $K$ with position. If the thermal properties depend on the temperature, the situation is more complicated since the equation becomes non-linear: few such cases have been studied in connexion with conduction of heat since the variation of the thermal properties with temperature is relatively slow and the information available about it is scanty and inaccurate. Nevertheless, they are becoming increasingly important when large ranges of temperature are involved, as in the solidification of castings; also the same equations arise in the theory of diffusion, where, because of the more rapid variation of diffusion coefficients with concentration, they are of much greater importance.† In most cases numerical methods have to be used, but a few general results, and cases in which exact solution is possible, will be noted below.

I. *The case of thermal properties varying with the temperature but independent of position*‡

In this case (8) becomes

$$\rho c \frac{\partial v}{\partial t} = K\nabla^2 v + A + \frac{\partial K}{\partial v}\left\{\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right\}, \tag{9}$$

which shows the non-linearity clearly.

† Cf. *M.D.*, Chaps. IX–XI.

‡ Some other methods for the one-dimensional case are given in § 2.16.

(8) may be reduced to a simpler form† by introducing the new variable

$$\Theta = \frac{1}{K_0}\int\limits_0^v K\,dv, \tag{10}$$

where $K_0$ is the value of $K$ when $v = 0$. This, and the lower limit of integration, are merely introduced to give $\Theta$ the dimensions of temperature and a definite value.

It follows from (10) that‡

$$\frac{\partial\Theta}{\partial t} = \frac{K}{K_0}\frac{\partial v}{\partial t}, \qquad \frac{\partial\Theta}{\partial x} = \frac{K}{K_0}\frac{\partial v}{\partial x}, \qquad \frac{\partial\Theta}{\partial y} = \frac{K}{K_0}\frac{\partial v}{\partial y}, \qquad \frac{\partial\Theta}{\partial z} = \frac{K}{K_0}\frac{\partial v}{\partial z},$$

and (8) becomes

$$\nabla^2\Theta - \frac{1}{\kappa}\frac{\partial\Theta}{\partial t} = -\frac{A}{K_0}, \tag{11}$$

where, in (11), $A$ and $\kappa = K/\rho c$ are expressed as functions of the new variable $\Theta$. Thus, in terms of this new variable, the form (7) of the equation of conduction of heat is preserved, but with a diffusivity $\kappa$ which depends on $\Theta$. It is a fact that in many cases the variation of $\kappa$ with temperature is much less important than that of $K$, so that, to a reasonable approximation, it may be taken to be constant; for example, for metals near absolute zero, both $K$ and $c$ are approximately proportional to the absolute temperature. In such cases, if $A$ is independent of $v$, equation (11) becomes of type (7) and solutions for the case of constant conductivity may be taken over immediately by replacing $v$ by $\Theta$, provided that the boundary conditions prescribe only $v$ or $K\,\partial v/\partial n$: if they are of the form

$$(\partial v/\partial n) + hv = 0,$$

where $h$ is a constant, this remark does not hold.

The case of steady flow is of particular importance since (11) becomes Poisson's equation if $A$ is constant, or Laplace's equation if $A = 0$. Thus solutions of problems of steady heat flow with conductivity any function of the temperature, and boundary conditions consisting of prescribed temperature or flux, may be derived immediately from the corresponding solutions for constant conductivity.

Another useful form may be obtained by introducing $W$, the heat content per unit mass of the material (measured from some arbitrary zero of temperature). In terms of this quantity, (8) becomes

$$\rho\frac{\partial W}{\partial t} = \frac{\partial}{\partial x}\left(K\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial v}{\partial z}\right) + A, \tag{12}$$

or, in terms of $\Theta$ defined by (10),

$$\frac{\rho}{K_0}\frac{\partial W}{\partial t} = \nabla^2\Theta + \frac{A}{K_0}, \tag{13}$$

where $W$ is related to $\Theta$ in a known manner. The introduction of $W$ has advantages in problems involving latent heat.

II. *Heat production in the solid*

Cases in which heat is produced in the solid are becoming increasingly important in technical applications. Heat may be produced by (i) the passage of an electric

† van Dusen, *Bur. Stand. J. Res.* **4** (1930) 753–6; Eyres, Hartree, Ingham, Jackson, Sarjant, and Wagstaff, *Phil. Trans. Roy. Soc.* A, **240** (1946) 1–58. For steady flow, the method dates back to Kirchhoff's *Vorlesungen über die Theorie der Wärme* (1894).

‡ $\Theta$ is thus essentially a potential whose gradient is proportional to the flux, cf. Vernotte, *Comptes Rendus*, **218** (1944) 39–41.