



# Applied Differential Games

Alexander Mehlmann





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Alexander Mehlmann

*University of Technology  
Vienna, Austria*

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## Preface

This book grew out of a set of lecture notes for a one-semester course on dynamic game theory held at the University of Technology, Vienna. It is intended primarily at the graduate level for students in operations research, management science, applied mathematics, and economics. I hope that I have been able to give the reader an accessible introduction to the subject of nonzero-sum differential games with particular emphasis on applications.

It would be irrational to try to reach total comprehensiveness in a single volume. Therefore, I have resisted the temptation to "over-cannibalize" previous textbooks and monographs on the subject. It has rather been my desire to cover material that (I think) is important and interesting, but gets left out of these publications.

Writing a book is quite a game. In the beginning—before closing the *binding agreement*\* with Plenum—I believed this to be a *finite horizon* game. Time, however,

\* Key words will be explained in the text.

was a merciless arbiter. I am grateful to the Senior Editor, Dr. Ken Derham, for allowing manuscript delivery to become a (restricted) *free terminal time* problem.

Most of all, I thank my wife Grace for offering me the needed spiritual support, and my two-year-old daughter Sabrina for ignoring the paradoxical situation that there are games which prevent Dad from playing with her.

The Department of Operations Research (under Professor Gustav Feichtinger's mild *Stackelberg leadership*) ensured *playability* of the game's actions. Access to *information* has been made possible by a grant (S 3204) from the Austrian Science Foundation.

In reading many preliminary versions, Gerhard Sorger provided an invaluable and *perfect feedback* of the state of the game. I am also grateful to my student Mr. Horst Holzweber who (by producing the drawings) made the game illustrative. Richard F. Hartl taught me that we are living in a *tragedy of the commons* world (by simultaneously using scarce resources such as paper, printer, and personal computers).

During the last stages of manuscript preparation, I deliberately engaged in an extreme version of a *zero sum differential game*—*kendo*, the art of Japanese fencing; hopefully an *efficient* and *credible threat* in any future game of refereeing.

Alexander Mehlmann

Vienna

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# 1

## Introduction

**Everything has been thought of before, but  
the problem is to think of it again.**

**J.W. von Goethe**



In this short chapter, we shall explain what is meant by differential games and sketch a history of this essential branch of game theory. Despite its convincing success in applied modeling, the discipline of differential games has gained a rather controversial reception in game theory circles. This is mainly due to the control-theory pronunciation of classical game theory vocables, apparently a violation of rules according to which some scientists play the game of game theory.

In the first part of the book, we will consequently prove that an accurate presentation of differential game concepts can be realized without neglecting basic game-theoretic rules of strategic interaction. The main emphasis, however, will be placed on the important features of dynamic informational play, which may be detected even after reduction of the differential game description to normal form. We shall furthermore illustrate how these dynamic properties may be translated into control-theoretic terms.

### **1.1. The Glass Bead Game**

Hermann Hesse's "The Glass Bead Game" appeared in Switzerland in 1943. The conspicuous parallels between this literary fiction of "a game of thoughts called the Glass

Bead Game" and the theory of games,<sup>(1)</sup> born in 1944, can be taken as evidence that the times were ripe for the scientific and intellectual evolution of a discipline able to incorporate significant values of our culture.

Von Neumann and Morgenstern's<sup>(1)</sup> construct lacked the classical mathematical achievements in the spirit of Newton—a language of symbols in which science and art could play an equivalent role. It was Isaacs<sup>(2)</sup> who thought of such effective tools: differential equations. His optimal pursuit and evasion schemes represented a variation of an old theme, played once by Leonardo da Vinci.

The applications Isaacs had in mind were mainly combat problems. By its very nature the mathematics of (zero-sum) differential games seemed to restrict the Glass Bead Game to matters of martial arts.

### *1.1.1. Isaacs's Decade*

The origins of differential game theory can be traced back to the late 1940s. Rufus Isaacs, then on appointment with Rand, formulated missile versus enemy aircraft pursuit schemes in terms of descriptive and navigation (state and control) variables, and found the main principle: the *tenet of transition*.

Further innovations in differential game techniques and basic concepts followed, but due to lack of financial support Isaacs's work did not appear in print until 1965. In the meantime the subject of control theory had reached its maturity; the *principle of optimality*—a one-player special case of the *tenet of transition*—had been put on the map.

Research in differential games seemed to concentrate