S I Hariharan & T H Moulden (Editors)

The University of Tennessee Space Institute

Numerical methods for partial differential equations

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PREFACE

The material contained herein represents the substance of a Short Course offered at The University of Tennessee Space Institute in March 1985. As such it constitutes a review of several important areas of research in the field of numerical approximation to partial differential equations. One of the concepts involved with the evolution of this course was the need to introduce the technical community to the latest developments in the numerical analysis of partial differential equations. To this end it seemed desirable to have the course organized by a mathematician (SIH) and an engineer (THM). We hope that this mix of disciplines will lead to benefits for both.

Many of the example problems selected for discussion herein have, as their foundation, the equations of continuum mechanics. These equations, and their physical background, are reviewed in an appendix. Chapter 1 presents a detailed discussion of elliptic differential equations approximated in the context of the finite element method. Chapter 2 discusses numerical solution techniques for purely hyperbolic problems while chapter 3 deals with the problems and numerical treatment for equations of mixed type. Chapter 4 is devoted to a discussion of the non-linear theory of elasticity and the propagation of slip lines. Chapters 5, 6, 7 and 8 are concentrated reviews of specific problem areas. Thus, chapter 5 is concerned with the details of shock capturing in gasdynamic problems, chapter 6 treats topics in the theory of absorbing boundary conditions for exterior problems, chapter 7 reviews the theory of spectral methods and discusses their application to non-linear gasdynamic problems while chapter 8 considers the use of finite element methods in the numerical solution of the Navier Stokes equations.

The Short Course Program at UTSI is managed by Ms. S. Shankle and falls under the general Academic Program of The Institute (directed by Dr. A. A. Mason, Associate Dean). This publication would not have been possible without the many long hours of devotion by Linda Hall and Pat Allen in typsetting using the TEX82 system on the UTSI VAX 11 Computer. To them our greatest thanks.

S. I. Hariharan
T. H. Moulden

SOME STANDARD NOTATION

Vectors and operators are printed in **bold face type** while sets and function spaces are denoted by $\mathcal{CALIGRAPHIC}$ type. As far as possible. the following symbols are standard:

$\mathbf{E}$	strain tensor
$\mathbf{m}$	mass flux vector
$\mathbf{n}$	normal vector
$\mathbf{T}$	stress tensor
$\mathbf{u}$	displacement vector
$\mathbf{u}, \mathbf{w}$	dummy variables
$\mathbf{v}$	velocity vector
$\mathbf{x}, \boldsymbol{\xi}$	position vectors
$\boldsymbol{\varsigma}$	vorticity vector
ν	kinematic viscosity
μ, λ	Lamé constants

(v_i) will be the components of the vector $\mathbf{v}$ in the selected basis.

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PART I

ELLIPTIC EQUATIONS

CHAPTER 1

NUMERICAL SOLUTION OF ELLIPTIC BOUNDARY VALUE PROBLEMS

George J. Fix*

1.1. Overview and Examples

The most frequently encountered elliptic boundary value problem is the Dirichlet problem for the Poisson equation:

$$\Delta\varphi = f \text{ in } \Omega \tag{1.1.1}$$

$$\varphi = g \text{ on } \partial\Omega \tag{1.1.2}$$

Here f and g are given functions and Ω is a region in $\mathcal{R}^n$ having $\partial\Omega$ as its boundary, and Δ denotes the Laplacian operator. In this chapter attention will be confined to planar regions ($n = 2$) and to the three dimensional case ($n = 3$). While most of the ideas introduced here can be used for ordinary differential equations ($n = 1$), techniques specialized for those equations are typically more efficient.

The Poisson equation arises in many contexts, and φ typically is a potential for a field variable. This for example is the case for incompressible potential flows where

$$\mathbf{v} = grad(\varphi) \tag{1.1.3}$$

is the fluid velocity [1]. In addition to (1.1.2) one can also specify Neumann conditions

$$\partial\varphi/\partial n = v, \tag{1.1.4}$$

n being the outer normal to Ω, or a mixture of (1.1.2) and (1.1.4) can be applied. For equation (1.1.4), compatibility implies

$$\int_\Omega f = \int_{\partial\Omega} v. \tag{1.1.5}$$

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A generalization of (1.1.1) is provided by the nonlinear equation

$$div\,[\rho(grad(\varphi))grad(\varphi)] = f, \tag{1.1.6}$$

where $\rho(\cdot)$ is a given function. This equation arises in the theory of potential compressible flows [2], when $\rho(\cdot)$ is the density. In this context Bernoulli's equation gives a functional relation between ρ and the fluid velocity (1.1.3). This equation is elliptic exactly when

$$\frac{d}{d\xi}\,[\rho(\xi)\xi] > 0, \tag{1.1.7}$$

a condition that is equivalent to the flow being subsonic [1].

An example involving higher order derivatives is provided by the biharmonic equation:

$$\Delta^2\varphi = f \tag{1.1.8}$$

$$\varphi = g \tag{1.1.9}$$

$$\partial\varphi/\partial n = v \tag{1.1.10}$$

Versions of this equation arise in both solid and fluid mechanics [1],[3].

The above are examples of what are called strongly elliptic problems. This term will be given a precise meaning in Section 1.2.1. Partial Differential equations of this type are generally thought to be the most tractable of all as far as numerical approximation is concerned. There are a wide variety of methods that have been successfully used in practice, and in fact there are even commercial computer codes for these problems which are advertised as "black boxes." Nevertheless, some caution should be given about taking this point of view too far. In this regard, the following points are relevant:

1. Special properties of the Poisson equation.
2. Complications introduced by a highly curved boundary $\partial\Omega$.
3. Complications introduced by singularities. These can arise from "corners" in the boundary $\partial\Omega$, or by pathological behavior of the data (e.g., the functions f and g in (1.1.1)–(1.1.2)).

In the literature on the numerical solution of elliptic boundary value problems one often sees an overemphasis on the Poisson equation (or closely related equations). While strongly elliptic problems are exceptional among partial differential equations, the Poisson equation is exceptional even within the class of strongly elliptic equations. It has properties that are not shared by other strongly elliptic operators. These properties can be expressed in different ways but they all reduce to the existence of a *maximum principle* [4]. The reader will recall that if φ satisfies the Poisson equation with homogeneous data

$$\Delta\varphi = 0 \text{ in } \Omega \tag{1.1.11}$$

(i.e., φ is harmonic), then the maximum of φ must occur on the boundary $\partial\Omega$:

$$\max_{\tilde{x}\in\Omega} |\varphi(\tilde{x})| \leq \max_{\underline{x}\in\partial D} |\varphi(\tilde{x})| \tag{1.1.12}$$

A similar result is true for minimum values.

In some numerical approximations the importance of (1.1.12) (and its discrete analogs) is totally transparent. This for example is the case in finite difference approximations, where (1.1.12) serves as both a powerful technical tool in error analysis as well as being responsible for desirable matrix properties like semi diagonal dominance [5]. In finite elements approximations, on the other hand, the role of (1.1.12) is less transparent. With many higher order elements. for example, the identification of a discrete analog of (1.1.12) is still an open problem. Moreover, the error analysis for finite element approximations does not directly use (1.1.12), and tends to be similar in structure to the analysis of other strongly elliptic problems. Nevertheless, if one looks deeply into finite element approximations of the Poisson equation - particularly the subtle yet important features of the algebraic systems produced - the favorable features generated by the existence of a maximum principle tend to emerge. The point to be made here is the performance of a particular numerical method when it is applied to the Poisson equation, may not be a totally accurate picture of how the method will behave for other strongly elliptic problems.

The second issue raised above concerns complicated geometrical regions Ω that arise in applications. Some methods are especially designed for simple regions (e.g., rectangles in $\mathcal{R}^2$ or cubes in $\mathcal{R}^3$), and are extremely efficient in these contexts. While various gimmicks can be used to extend them beyond simple regions, their efficiency typically deteriorates in the process. On the other hand, there are methods which can be applied to very general regions Ω, but which are very inefficient when compared to the specialized methods for problems in simple regions. The point is that the decision on which approach to use in any given application is not automatic, and requires numerical analysis.

The third and final issue cited above concerns singularities. As noted earlier these can arise either through "defects" (e.g. corners) in the region Ω or through singularities in the data or coefficients in the equations. Not all methods can perform efficiently in the presence of singularities. Moreover, even for those methods which can deal with singularities (by for example grid refinement or special singular functions) there is the issue of adapting these techniques to the special features of specific applications. Impressive strides have been made in automatic self-adapting numerical approximation [6], but these are generally built around "global error criteria." In many applications, only special functionals of the solution are important, and adapting to these criteria typically requires additional numerical analysis.

In summary, it is perhaps valid to assert that strongly elliptic boundary value problems are the "easiest" of all problems involving the numerical approximation of partial differential equation. However, it would be a distortion to assert that their solution is routine.

Section 1.2 of this chapter is devoted to a survey of strongly elliptic problems and this is followed in Section 1.3 with a discussion of what are sometimes called weakly elliptic problems. The latter will be designated as elliptic problems of the stationary type for reasons that will be discussed in the text. The latter can be readily distinguished from strongly elliptic operators. First, they are generally "hard" from the numerical point of view. There are "reasonable" approaches which are totally unstable when applied to these problems This is in striking contrast to the strongly elliptic case where just about all of the "reasonable" approaches work. There may be more efficient alternatives, but the results are rarely catastrophic.

A second feature of the stationary elliptic case is that most of the technologically significant applications involving elliptic equations tend to be of this type. This is particularly true in fluid mechanics, and we end Section 1.1 by citing some specific examples.

The first example is provided by the Navier-Stokes equation for an incompressible fluid [7]:

$$\mathbf{v} \cdot grad(\mathbf{v}) - grad(p) = \nu\Delta\mathbf{v} \text{ in } \Omega \tag{1.1.13}$$

$$div(\mathbf{v}) = 0 \text{ in } \Omega \tag{1.1.14}$$

$$\mathbf{v} = \mathbf{V} \text{ on } \partial\Omega \tag{1.1.15}$$

In (1.1.13), $\nu > 0$ stands for the kinematic viscosity, $\mathbf{v}$ is the fluid velocity, and p is the pressure. The density has been normalized to unity. The boundary condition (1.1.15) is defined in terms of a given boundary velocity $\mathbf{V}$. These equations stand midway between the strongly elliptic and stationary cases. For numerical approximations that are capable of working in spaces of divergent free vector fields, these equations behave like a strongly elliptic system at least for sufficiently small Reynolds numbers (i.e., large ν). To treat (1.1.13) - (1.1.15) directly, on the other hand, leads to a stationary type of scheme. This can display deleterious instabilities which are unacceptable in most applications. There is a large literature on this subject which will be reviewed in Section 1.3.

A second example is provided by the two dimensional stream function - vorticity formulation of the equations of fluid mechanics [8]:

$$J(\psi, \varsigma) = \nu\Delta\varsigma \text{ in } \Omega \tag{1.1.16}$$

$$\Delta\psi = \varsigma \text{ in } \Omega \tag{1.1.17}$$

$$\psi = g \text{ on } \partial\Omega \tag{1.1.18}$$

$$\partial\psi/\partial n = h \text{ on } \partial\Omega \tag{1.1.19}$$

In (1.1.16), $J(\cdot,\cdot)$ is a bilinear form defined by

$$J(\psi,\varsigma) = \frac{\partial\psi}{\partial x_1}\frac{\partial\varsigma}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\frac{\partial\varsigma}{\partial x_1} \tag{1.1.20}$$

As in (1.1.13), $\nu > 0$ denotes the viscosity while ψ is the stream function and ς is the vorticity. Equations (1.1.17) and (1.1.18) can be combined to give a fourth order equation

$$J(\psi, \Delta\psi) = \nu\Delta^2\psi \tag{1.1.21}$$

for the stream function. For sufficiently large ν, (1.1.21) is a strongly elliptic system. However, attempts to deal with (1.1.16) to (1.1.19) directly leads to schemes of the stationary type ([9]-[10]).

Our final example is drawn from steady inviscid compressible flows. In standard (nonconservative form) these equations can be written (using the notation of (1.1.13) to (1.1.15))

$$\mathbf{v} \cdot grad(\mathbf{v}) - grad(p)/\rho = 0 \tag{1.1.22}$$

$$div(\rho\mathbf{v}) = 0 \tag{1.1.23}$$

$$div(\rho\mathbf{v}E) = 0 \tag{1.1.24}$$

Here

$$E = I + \frac{1}{2}\mathbf{v} \cdot \mathbf{v} \tag{1.1.25}$$

is the energy with I being the internal energy. The equation of state gives the functional relation

$$p = p(\rho, I), \tag{1.1.26}$$

and suitable boundary conditions must be adjoined to (1.1.22) - (1.1.24). The sound speed is defined through

$$c^2 = \partial p/\partial \rho, \tag{1.1.27}$$

and for subsonic flows; i.e.,

$$c > |\mathbf{v}|, \tag{1.1.28}$$

this system is elliptic. This is perhaps the most challenging of all elliptic systems and great care must be taken with numerical approximations as will be discussed in Section 1.3. One can add viscous terms [8] to (1.1.22)-(1.1.24), and from the numerical point of view these terms are denizen. In many applications the conservative form of the equations is preferred. This is obtained by rewriting (1.1.22) and using special combinations of (1.1.23)-(1.1.24) to give a system of the form

$$div(\mathbf{F}) = 0$$

for a suitable vector $\mathbf{F}$ as in reference [8].

Certain topics had to be omitted from this chapter in order to keep it to a reasonable length. One of the most regrettable omissions is the subject of integral equations, and in particular the boundary integral method. The reader is referred to the excellent review article in reference [11] as well as reference [12] for a discussion of these methods.

1.2 Strongly Elliptic Problems

1.2.1 Basic Properties

The philosophical viewpoint adopted throughout the subject of the numerical solution of partial differential equations has been to view the partial differential equation as an operator equation

$$L\varphi = f \tag{1.2.1}$$

The next step is to identify spaces S_1, S_2 for which the inverse L^{-1} of L is continuous

$$L^{-1} : S_2 \longrightarrow S_1 \tag{1.2.2}$$

From this point of view it is taken as an article of faith in numerical analysis that continuous operations can be appropriately discretized to yield stable and convergent approximations.

This point of view gets lost in most treatments of the numerical solution of elliptic equations due to the ubiquitous nature of $\mathcal{L}^2$ spaces based on the inner product

$$(\varphi, \psi) = \int_{\Omega} \varphi\psi \tag{1.2.3}$$

Indeed, for most elliptic problems (both those which are strongly elliptic as well as those of the stationary type) $\mathcal{L}^2$ spaces and their derivatives are entirely appropriate and satisfy the continuity requirements cited above. Exception occur in problems having singularities. In the latter case, weighted $\mathcal{L}^2$ spaces associated with inner products of the form

$$(\varphi, \psi)_\sigma = \int \sigma\varphi\psi \tag{1.2.4}$$

for an appropriate weighting function σ arise in a natural way. In the first part of this chapter attention will be confined to spaces based on equation (1.2.3). The treatment of weighted spaces will be given in section 1.3.

In order to define what is meant by a strongly elliptic equation one must first identify the $\mathcal{L}^2$ -form of the equation along with the associated *spaces of test and trial functions*. We do this by the *Galerkin process* (associated with $\mathcal{L}^2$). Before giving a general definition let us first illustrate the process with the following example:

$$-div(\rho\, grad(\varphi)) = f \quad \text{in } \Omega \tag{1.2.5}$$

$$\varphi = 0 \quad \text{on } \Gamma_1 \tag{1.2.6}$$

$$\partial\varphi/\partial n = 0 \quad \text{on } \Gamma_2 \tag{1.2.7}$$

Here the boundary Ω has two parts, namely Γ_1 where a Dirichlet condition is specified and Γ_2 where a Newmann condition is specified. The function ρ depends on grad(φ) as in the example in section 1.1. The first step is to multiply equation (1.1.5) by a *test function* ψ and integrate on Ω. (This is where we arbitrarily introduce the $\mathcal{L}^2$ space into the problem).

$$-\int_\Omega \psi div(\rho\, grad(\varphi)) = \int_\Omega \psi f \tag{1.2.8}$$

an integration by parts gives

$$-\int_{\partial\Omega} \varphi\rho\partial\psi/\partial n + \int_\Omega \rho\, grad(\psi) grad(\varphi) = \int_\Omega \psi f \tag{1.2.9}$$

The boundary integral on the left consists of two terms: One over Γ_1 and the other over Γ_2. The latter is zero because of (1.2.7), and by requiring

$$\psi = 0 \text{ on } \Gamma_1, \tag{1.2.10}$$