TABLE OF INTEGRALS, SERIES, AND PRODUCTS

CORRECTED AND ENLARGED EDITION

I. S. Gradshteyn/I. M. Ryzhik

Table of Integrals, Series, and Products

I. S. GRADSHTEYN and I. M. RYZHIK

Corrected and Enlarged Edition
Prepared by

Alan Jeffrey
PROFESSOR OF ENGINEERING MATHEMATICS
UNIVERSITY OF NEWCASTLE UPON TYNE, ENGLAND

Incorporating the fourth edition prepared by
Yu. V. Geronimus and M. Yu. Tseytlin

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PREFACE TO THE CORRECTED AND ENLARGED EDITION

When this book was originally planned by Ryzhik, it was designed to assist mathematicians, scientists, and engineers, who at that time relied almost entirely on analytical solutions to their problems. These were usually obtained in complicated series form involving special functions, and for this purpose a comprehensive table of integrals was necessary. The success of this basic plan was amply demonstrated by the fact that by the time the English language translation was published in 1965, the original Russian version had gained Gradshteyn as a coauthor and evolved through four separate editions to become a book of 1080 pages.

After the appearance of the translation, the book quickly established its position as one of the major reference sources for integrals available in the English language. However, just as during its early years the book changed its structure and content to meet the needs of its Russian users, so also must the English language version adapt to the needs of its new users. With the present-day ready availability of powerful computers, the necessity for cumbersome series-type solutions has all but been eliminated. In their place have come other types of analytical problems of equivalent complexity, some of a traditional nature, and some quite new and often arising from numerical analysis. They have in common the fact that they also make a demand on tables of this type, showing yet again the fundamental soundness of the original plan.

Because of the increasing complexity of problems, which often makes an analytical solution impossible, there is now a growing need to estimate properties of solutions rather than to find explicit solutions. It was with these ideas in mind that this extended edition was prepared, containing the new material of Sections 10 to 17. Apart from the inclusion of some basic results on vector operators and coordinate systems that is likely to be useful during the formulation of many problems, most of the new material is concerned with inequalities. These range from fundamental algebraic and functional inequalities to integral inequalities and basic oscillation and comparison theorems for ordinary differential equations. The continuing important part played by integral transforms has made it desirable to include basic reference lists of transform pairs for the Fourier and Laplace transforms in the same volume that provides so much information about other integrals.

It is hoped that these major additions to the body of the tables are in the spirit of the original plan of the book, and that they will still further enhance its value. All known errors in any one printing of the book have been corrected at the subsequent reprinting, so that this present edition incorporates all corrections made since 1965. I am extremely grateful to the many people named in the acknowledgment on page ix, who over the years have helped me to free this work from errors.

A. Jeffrey

PREFACE TO THE FIRST EDITION

The number of formulas for integrals, sums, series, and products available in the existing mathematical reference books is definitely insufficient for mathematicians working in scientific fields and for scientists and engineers doing theoretical and experimental research. The present tables have been compiled with the purpose of filling this gap. More than five thousand formulas have been collected in the present volume from various sources.

The book is primarily intended for scientists, technicians, and engineers carrying out investigations in the physical-mathematical sciences. Therefore, exposition takes up only a small portion of the book, and basically, it is a collection of formulas.

Considerable attention has been given to special functions, in particular, elliptic, Bessel, and Legendre functions. Many formulas dealing with these functions are included.

I wish to take this occasion to express my deep gratitude to Professors V. V. Stepanov, A. I. Markushevich, and I. N. Bronshteyn for valuable advice and for suggestions made by them in the preparation of this work.

I. Ryzhik

PREFACE TO THE THIRD EDITION

I. M. Ryzhik, the author of the first and second editions of these tables, died during World War II. These tables have been considerably revised in preparation for printing.

All the formulas, definitions, and theorems have been placed into numbered divisions. The numbering procedure has some similarity with the decimal system of classification and can easily be understood from the arrangement of the table of contents. Only the main divisions are included in the table of contents. These have numbers containing one, two, or three digits. The smallest divisions of the book are numbered with four digits. These divisions contain one or several formulas, theorems, or definitions, which are numbered in light type with successive integers. The digit 0 is reserved for divisions of a general character: introductions, definitions, etc. The number 0 is also used for the first chapter of the book, which includes a number of theorems of a general nature and which is itself something of an introduction.

I. Gradshteyn

PREFACE TO THE FOURTH EDITION

In preparing the fourth edition, I. S. Gradshteyn intended to expand this reference work considerably. His death prevented completion of this plan. He had compiled new tables of integrals of the elementary functions and had collected material for preparing tables of integrals of special functions.

The publishing authorities entrusted us with preparing the manuscript left by Gradshteyn and with adding supplementary material where it was needed.

In carrying out this work, we have tried to follow the plan of the manuscript and of the preceding edition. Throughout, we have kept its main feature: the order of the formulas. We have included the divisions dealing with sums, series, products, and elementary functions without change from the preceding edition. The other divisions were somewhat revised. The tables of definite integrals of elementary and special functions, in particular, were considerably extended. We have added a few new divisions: the integrals of Mathieu functions, Struve functions, Lommel functions, and a number of other functions that were entirely omitted from the older editions. In general, the number of special functions considered in the fourth edition is much greater than in the third. As a consequence, the chapters dealing with special functions were supplemented with the corresponding divisions.

The majority of definitions of special functions that were used in the preceding edition have been kept. We changed to new definitions only in certain cases when we were following references that contain the major source of material on integrals of the special functions in question.

Certain notations have also been changed. The chapter in the third edition that is devoted to integral transformations was dropped from the fourth edition. The material included in it appears in other portions of the book.

We wish to express our deep gratitude to A. F. Lapko, who read the manuscript carefully and made a number of useful comments.

Yu. Geronimus, M. Tseytlin

EDITOR'S FOREWORD

This book is a translation of the fourth and most recent edition of Ryzhik and Gradshteyn's *Table of Integrals*. It represents a very considerable enlargement of the third edition to which many new sections have been added. Its thorough coverage of the functions of mathematical physics and its careful arrangement of material will make it invaluable to mathematicians, engineers, and physicists working in all fields.

In preparing this translation the opportunity has been taken to add a short note on the use of the tables. After a brief reference to the method of arrangement of the tables, the introduction then summarizes some of the more common differences in notation and definition of higher transcendental functions. A short classified bibliography has also been added at the end of the book to supplement the original Russian bibliography—much of which is likely to be inaccessible to the user of these tables.

A. Jeffrey

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THE ORDER OF PRESENTATION OF THE FORMULAS

The question of the most expedient order in which to give the formulas, in particular, in what division to include particular formulas such as the definite integrals, turned out to be quite complicated. The thought naturally occurs to set up an order analogous to that of a dictionary. However, it is almost impossible to create such a system for the formulas of integral calculus. Indeed, in an arbitrary formula of the form

$$\int_a^b f(x)\,dx = A$$

one may make a large number of substitutions of the form $x=\varphi(t)$ and thus obtain a number of "synonyms" of the given formula. We must point out that the table of definite integrals by Bierens de Haan and the earlier editions of the present reference both sin in the plethora of such "synonyms" and the formulas of complicated form. In the present edition, we have tried to keep only the simplest of the "synonym" formulas. Basically, we judged the simplicity of a formula from the standpoint of the simplicity of the arguments of the "outer" functions that appear in the integrand. Where possible, we have replaced a complicated formula with a simpler one. Sometimes, several complicated formulas were thereby reduced to a single simpler one. We then kept only the simplest formula. As a result of such substitutions, we sometimes obtained an integral that could be evaluated by use of the formulas of chapter two and the Newton-Leibnitz formula, or to an integral of the form

$$\int_{-a}^{a} f(x)\,dx,$$

where $f(x)$ is an odd function. In such cases the complicated integrals have been omitted.

Let us give an example using the expression

$$\int_0^{\frac{\pi}{4}} \frac{(\operatorname{ctg} x-1)^{p-1}}{\sin^2 x} \ln \operatorname{tg} x\,dx = -\frac{\pi}{p}\operatorname{cosec} p\pi. \qquad (1)$$

By making the natural substitution $\operatorname{ctg} x - 1 = u$, we obtain

$$\int_0^{\infty} u^{p-1} \ln(1+u)\,du = \frac{\pi}{p}\operatorname{cosec} p\pi. \qquad (2)$$

Integrals similar to formula (1) are omitted in this new edition. Instead, we have formula (2) and the formula obtained from the integral (1) by making the substitution $\operatorname{ctg} x = v$.

As a second example, let us take

$$I = \int_0^{\frac{\pi}{2}} \ln(\operatorname{tg}^p x + \operatorname{ctg}^p x) \ln \operatorname{tg} x \, dx = 0.$$

The substitution $\operatorname{tg} x = u$ yields

$$I = \int_0^{\infty} \frac{\ln(u^p + u^{-p}) \ln u}{1 + u^2} \, du.$$

If we now set $v = \ln u$, we obtain

$$I = \int_{-\infty}^{\infty} \frac{ve^v}{1 + e^{2v}} \ln(e^{pv} + e^{-pv}) \, dv = \int_{-\infty}^{\infty} v \frac{\ln 2 \operatorname{ch} pv}{2 \operatorname{ch} v} \, dv.$$

The integrand is odd and, consequently, the integral is equal to 0.

Thus, before looking for an integral in the tables, the user should simplify as much as possible the arguments (the "inner" functions) of the functions in the integrand.

The functions are ordered as follows:

First we have the elementary functions:

1. The function $f(x) = x$.
2. The exponential function.
3. The hyperbolic functions.
4. The trigonometric functions.
5. The logarithmic function.
6. The inverse hyperbolic functions. (These are replaced with the corresponding logarithms in the formulas containing definite integrals.)
7. The inverse trigonometric functions.

Then follow the special functions:

8. Elliptic integrals.
9. Elliptic functions.
10. The logarithm integral, the exponential integral, the sine integral, and the cosine integral functions.
11. Probability integrals and Fresnel's integrals.
12. The gamma function and related functions.
13. Bessel functions.
14. Mathieu functions.
15. Legendre functions.
16. Orthogonal polynomials.
17. Hypergeometric functions.
18. Degenerate hypergeometric functions.
19. Functions of a parabolic cylinder.
20. Meijer's and MacRobert's functions.
21. Riemann's zeta function.

The integrals are arranged in order of outer function according to the above scheme: the farther down in the list a function occurs, (i.e. the more complex it is) the later will the corresponding formula appear in the tables. Suppose that several expressions have the same outer function. For example, consider $\sin e^x$, $\sin x$, $\sin \ln x$. Here, the outer function is the sine in all three cases. Such expressions are then arranged in order of the inner function. In the present work, these functions are therefore arranged in the following order: $\sin x$, $\sin e^x$, $\sin \ln x$.

Our list does not include polynomials, rational functions, powers, or other algebraic functions. An algebraic function that is included in tables of definite integrals can usually be reduced to a finite combination of roots of rational power. Therefore, for classifying our formulas, we can conditionally treat a power function as a generalization of an algebraic and, consequently, of a rational function.* We shall distinguish between all these functions and those listed above and we shall treat them as operators. Thus, in the expression $\sin^2 e^x$, we shall think of the squaring operator as applied to the outer function, namely, the sine. In the expression $\frac{\sin x + \cos x}{\sin x - \cos x}$, we shall think of the rational operator as applied to the trigonometric functions sine and cosine. We shall arrange the operators according to the following order:

1. Polynomials (listed in order of their degree).
2. Rational operators.
3. Algebraic operators (expressions of the form $A^{\frac{p}{q}}$, where q and p are rational, and $q > 0$; these are listed according to the size of q).
4. Power operators.

Expressions with the same outer and inner functions are arranged in the order of complexity of the operators. For example, the following functions (whose outer functions are all trigonometric, and whose inner functions are all $f(x) = x$) are arranged in the order shown:

$$\sin x,\ \sin x \cos x,\ \frac{1}{\sin x} = \operatorname{cosec} x,\ \frac{\sin x}{\cos x} = \operatorname{tg} x,\ \frac{\sin x + \cos x}{\sin x - \cos x},\ \sin^m x, \sin^m x \cos x.$$

Furthermore, if two outer functions $\varphi_1(x)$ and $\varphi_2(x)$, where $\varphi_1(x)$ is more complex than $\varphi_2(x)$, appear in an integrand and if any of the operations mentioned are performed on them, then the corresponding integral will appear (in the order determined by the position of $\varphi_2(x)$ in the list) after all integrals containing only the function $\varphi_1(x)$. Thus, following the trigonometric functions are the trigonometric and power functions (that is, $\varphi_2(x) = x$). Then come

combinations of trigonometric and exponential functions,

combinations of trigonometric functions, exponential functions, and powers, etc.,

combinations of trigonometric and hyperbolic functions, etc.

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*For any natural number n, the involution $(a+bx)^n$ of the binomial $a+bx$ is a polynomial. If n is a negative integer, $(a+bx)^n$ is a rational function. If n is irrational, the function $(a+bx)^n$ is not even an algebraic function.

Integrals containing two functions $\varphi_1(x)$ and $\varphi_2(x)$ are located in the division and order corresponding to the more complicated function of the two. However, if the positions of several integrals coincide because they contain the same complicated function, these integrals are put in the position defined by the complexity of the second function.

To these rules of a general nature, we need to add certain particular considerations that will be easily understood from the tables. For example, according to the above remarks, the function $e^{\frac{1}{x}}$ comes after e^x as regards complexity, but $\ln x$ and $\ln \frac{1}{x}$ are equally complex since $\ln \frac{1}{x} = -\ln x$. In the section on "powers and algebraic functions", polynomials, rational functions, and powers of powers are formed from power functions of the form $(a + bx)^n$ and $(\alpha + \beta x)^\nu$.

USE OF THE TABLES*

For the effective use of the tables contained in this book it is necessary that the user should first become familiar with the classification system for integrals devised by the authors Ryzhik and Gradshteyn. This classification is described in detail in the section entitled The Order of Presentation of the Formulas and essentially involves the separation of the integrand into *inner* and *outer* functions. The principal function involved in the integrand is called the *outer* function and its argument, which is itself usually another function, is called the *inner* function. Thus, if the integrand comprised the expression $\ln \sin x$, the *outer* function would be the logarithmic function while its argument, the *inner* function, would be the trigonometric function $\sin x$. The desired integral would then be found in the section dealing with logarithmic functions, its position within that section being determined by the position of the *inner* function (here a trigonometric function) in Ryzhik and Gradshteyn's list of functional forms.

It is inevitable that some duplication of symbols will occur within such a large collection of integrals and this happens most frequently in the first part of the book dealing with algebraic and trigonometric integrands. The symbols most frequently involved are α, β, γ, δ, t, u, z, z_k, and Δ. The expressions associated with these symbols are used consistently within each section and are defined at the start of each new section in which they occur. Consequently, reference should be made to the beginning of the section being used in order to verify the meaning of the substitutions involved.

Integrals of algebraic functions are expressed as combinations of roots with rational power indices, and definite integrals of such functions are frequently expressed in terms of the Legendre elliptic integrals $F(\phi, k)$, $E(\phi, k)$ and $\Pi(\phi, n, k)$, respectively, of the first, second and third kinds.

An abbreviated notation is used for the trigonometric and hyperbolic functions $\tan x$, $\cot x$, $\sinh x$, $\cosh x$, $\tanh x$ and $\coth x$ which are denoted, respectively, by $\operatorname{tg} x$, $\operatorname{ctg} x$, $\operatorname{sh} x$, $\operatorname{ch} x$, $\operatorname{th} x$ and $\operatorname{cth} x$. Also the four inverse hyperbolic functions $\operatorname{Arsh} z$, $\operatorname{Arch} z$, $\operatorname{Arth} z$ and $\operatorname{Arcth} z$ are introduced through the definitions

$$\arcsin z = \frac{1}{i} \operatorname{Arsh} (iz)$$

$$\arccos z = \frac{1}{i} \operatorname{Arch} z$$

$$\operatorname{arctg} z = \frac{1}{i} \operatorname{Arth} (iz)$$

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*Prepared by Alan Jeffrey for the English language edition.

$$\operatorname{arcctg} z = i \operatorname{Arcth}(iz)$$

or,

$$\operatorname{Arsh} z = \frac{1}{i} \arcsin(iz)$$

$$\operatorname{Arch} z = i \arccos z$$

$$\operatorname{Arth} z = \frac{1}{i} \operatorname{arctg}(iz)$$

$$\operatorname{Arcth} z = \frac{1}{i} \operatorname{arcctg}(-iz)$$

The numerical constants C and G which often appear in the definite integrals denote Euler's constant and Catalan's constant, respectively. Euler's constant C is defined by the limit

$$C = \lim_{s \to \infty} \left(\sum_{m=1}^{s} \frac{1}{m} - \ln s \right) = 0.577215\ldots\ .$$

On occasions other writers denote Euler's constant by the symbol γ, but this is also often used instead to denote the constant

$$\gamma = e^{C} = 1.781072\ldots\ .$$

Catalan's constant G is related to the complete elliptic integral

$$K \equiv K(k) \equiv \int_0^{\pi/2} \frac{da}{\sqrt{1 - k^2 \sin^2 a}}$$

by the expression

$$G = \frac{1}{2} \int_0^1 K\,dk = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} = 0.915965\ldots\ .$$

Since the notations and definitions for higher transcendental functions that are used by different authors are by no means uniform, it is advisable to check the definitions of the functions that occur in these tables. This can be done by identifying the required function by symbol and name in the Index of Special Functions and Notations that appears at the front of the book on page xxxix, and by then referring to the defining formula or section number listed there. We now present a brief discussion of some of the most commonly used alternative notations and definitions for higher transcendental functions.

Bernoulli and Euler Polynomials and Numbers

Extensive use is made throughout the book of the Bernoulli and Euler numbers B_n and E_n that are defined in terms of the Bernoulli and Euler polynomials of order n, $B_n(x)$ and $E_n(x)$, respectively. These polynomials are defined by the generating functions

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \qquad \text{for} \qquad |t| < 2\pi$$

and

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \qquad \text{for} \qquad |t| < \pi.$$

The Bernoulli numbers are always denoted by B_n and are defined by the relation

$$B_n = B_n(0) \qquad \text{for} \qquad n = 0, 1, \ldots ,$$

when

$$B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_2 = 1/6,\ B_4 = -1/30, \ldots .$$

The Euler numbers E_n are defined by setting

$$E_n = 2^n E_n\left(\frac{1}{2}\right) \qquad \text{for} \qquad n = 0, 1, \ldots .$$

The E_n are all integral and $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61, \ldots$.

An alternative definition of Bernoulli numbers, which we shall denote by the symbol B_n^*, uses the same generating function but identifies the B_n^* differently in the following manner:

$$\frac{t}{e^t - 1} = 1 - \frac{1}{2}t + B_1^*\frac{t^2}{2!} - B_2^*\frac{t^4}{4!} + \cdots .$$

This definition then gives rise to the alternative set of Bernoulli numbers

$B_1^* = 1/6$, $B_2^* = 1/30$, $B_3^* = 1/42$, $B_4^* = 1/30$, $B_5^* = 5/66$,
$B_6^* = 691/2730$, $B_7^* = 7/6$, $B_8^* = 3617/510, \ldots$.

These differences in notation must also be taken into account when using the following relationships that exist between the Bernoulli and Euler polynomials:

$$B_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} B_{n-k} E_k(2x) \qquad n = 0, 1, \ldots$$

$$E_{n-1}(x) = \frac{2^n}{n}\left\{B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right)\right\}$$

or

$$E_{n-1}(x) = \frac{2}{n}\left\{B_n(x) - 2^n B_n\left(\frac{x}{2}\right)\right\} \qquad n = 1,2,\ldots$$

and

$$E_{n-2}(x) = 2\binom{n}{2}^{-1} \sum_{k=0}^{n-2} \binom{n}{k} (2^{n-k} - 1) B_{n-k} B_k(x) \qquad n = 2,3,\ldots .$$

There are also alternative definitions of the Euler polynomial of order n, and it should be noted that some authors, using a modification of the third expression above, call

$$\left(\frac{2}{n+1}\right)\left\{B_n(x) - 2^n B_n\left(\frac{x}{2}\right)\right\}$$

the n-th Euler polynomial.

Elliptic Functions and Elliptic Integrals

The following notations are often used in connection with the inverse elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$ and $\operatorname{dn} u$:

$$\operatorname{ns} u = \frac{1}{\operatorname{sn} u} \qquad \operatorname{nc} u = \frac{1}{\operatorname{cn} u} \qquad \operatorname{nd} u = \frac{1}{\operatorname{dn} u}$$

$$\operatorname{sc} u = \frac{\operatorname{sn} u}{\operatorname{cn} u} \qquad \operatorname{cs} u = \frac{\operatorname{cn} u}{\operatorname{sn} u} \qquad \operatorname{ds} u = \frac{\operatorname{dn} u}{\operatorname{sn} u}$$

$$\operatorname{sd} u = \frac{\operatorname{sn} u}{\operatorname{dn} u} \qquad \operatorname{cd} u = \frac{\operatorname{cn} u}{\operatorname{dn} u} \qquad \operatorname{dc} u = \frac{\operatorname{dn} u}{\operatorname{cn} u}$$

The following elliptic integral of the third kind is defined by Ryzhik and Gradshteyn to be

$$\Pi(\varphi,n,k) = \int_0^{\varphi} \frac{da}{(1 + n\sin^2 a)\sqrt{1 - k^2\sin^2 a}} = \int_0^{\sin\varphi} \frac{dx}{(1 + nx^2)\sqrt{(1 - x^2)(1 - k^2x^2)}}.$$

Other authors use the definition

$$\Pi(\varphi,n^2,k) = \int_0^{\varphi} \frac{da}{(1 - n^2\sin^2 a)\sqrt{1 - k^2\sin^2 a}}$$

$$= \int_0^{\sin\varphi} \frac{dx}{(1 - n^2x^2)\sqrt{(1 - x^2)(1 - k^2x^2)}} \qquad (-\infty < n^2 < \infty).$$

The Jacobi Zeta Function and Theta Functions

The Jacobi zeta function zn (u,k), frequently written $Z(u)$, is defined by the relation

$$\text{zn}(u,k) = Z(u) = \int_0^u \left\{\text{dn}^2 v - \frac{E}{K}\right\} dv = E(u) - \frac{E}{K} u .$$

This is related to the theta functions by the relationship

$$\text{zn}(u,k) = \frac{\partial}{\partial u} \ln \Theta(u)$$

giving

$$\text{(i)} \quad \text{zn}(u,k) = \frac{\pi}{2K} \frac{\vartheta_1'\left(\frac{\pi u}{2K}\right)}{\vartheta_1\left(\frac{\pi u}{2K}\right)} - \frac{\text{cn}\, u\ \text{dn}\, u}{\text{sn}\, u}$$

$$\text{(ii)} \quad \text{zn}(u,k) = \frac{\pi}{2K} \frac{\vartheta_2'\left(\frac{\pi u}{2K}\right)}{\vartheta_2\left(\frac{\pi u}{2K}\right)} + \frac{\text{dn}\, u\ \text{sn}\, u}{\text{cn}\, u}$$

$$\text{(iii)} \quad \text{zn}(u.k) = \frac{\pi}{2K} \frac{\vartheta_3'\left(\frac{\pi u}{2K}\right)}{\vartheta_3\left(\frac{\pi u}{2K}\right)} - k^2 \frac{\text{sn}\, u\ \text{cn}\, u}{\text{dn}\, u}$$

$$\text{(iv)} \quad \text{zn}(u,k) = \frac{\pi}{2K} \frac{\vartheta_4'\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)} .$$

Many different notations for the theta function are in current use. The most common variants are the replacement of the argument u by the argument u/π and, occasionally, a permutation of the identification of the functions ϑ_1 to ϑ_4 with the function ϑ_4 replaced by ϑ.

The Factorial (Gamma) Function

In older reference texts the gamma function $\Gamma(z)$, defined by the Euler integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt ,$$

is sometimes expressed in the alternative notation

$$\Gamma(1 + z) = z! = \Pi(z).$$

On occasions the related derivative of the logarithmic factorial function $\Psi(z)$ is used where

$$\frac{d(\ln z!)}{dz} = \frac{(z!)'}{z!} = \Psi(z+1).$$

This function satisfies the recurrence relation

$$\Psi(z) = \Psi(z-1) + \frac{1}{z-1}$$

and is defined by the series

$$\Psi(z) = -C + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{z+n}\right).$$

The derivative $\Psi'(z)$ satisfies the recurrence relation

$$-\Psi'(z-1) = -\Psi'(z) + \frac{1}{z^2}$$

and is defined by the series

$$\Psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$

Exponential and Related Integrals

The exponential integrals $E_n(z)$ have been defined by Schloemilch using the integral

$$E_n(z) = \int_0^{\infty} e^{-zt}\, t^{-n}\, dt \qquad (n = 0, 1, \ldots, \operatorname{Re} z > 0).$$

They should not be confused with the Euler polynomials already mentioned. The function $E_1(z)$ is related to the exponential integral $\mathrm{Ei}(z)$ and to the logarithmic integral $\mathrm{li}(z)$ through the expressions

$$E_1(z) = -\mathrm{Ei}(-z) = \int_z^{\infty} e^{-t}\, t^{-1}\, dt$$