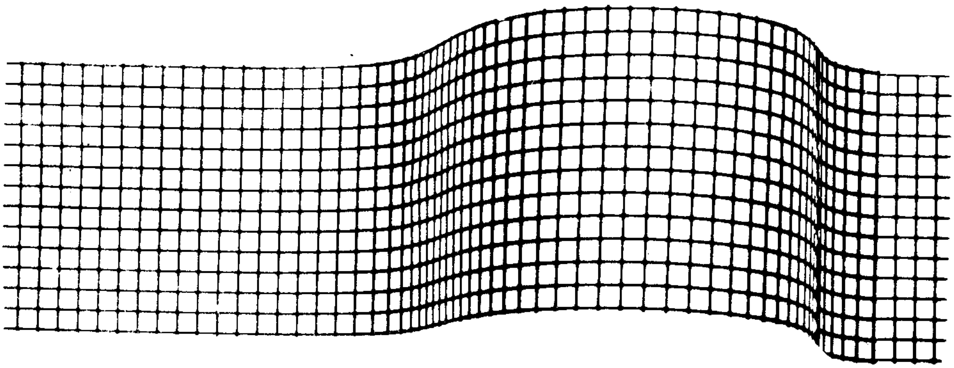
OPERATIONAL RESEARCH BY EXAMPLE

Colin Palmer and Alexander E. Innes



OPERATIONAL RESEARCH BY EXAMPLE

Colin F. Palmer
M.Sc., C.Eng., M.B.I.M., M.I.Prod.E.

Lecturer in Quantitative Methods
University of Birmingham

and

Alexander E. Innes
B.Sc. (Econ.) (Hons), D.P.A. (Lond.), Cert. Ed. (Leeds)

Senior Lecturer in Business Statistics
Liverpool Polytechnic

M

Preface

This book introduces the nature of *operational research* and some basic techniques to students in higher and further education and to people working in business and public administration. Explanation is mainly through carefully chosen examples and exercises, occasionally simplified, but otherwise intended to be realistic in context.

Most readers are likely to have the basic knowledge needed to follow the calculations given in the text and to attempt the exercises; those with no calculus or statistics should be able to master most of the material; and those without may find the two books by A. E. Innes mentioned on page ii helpful.

Examples and exercises have been kept fairly simple. Application of the principles they use to business and to administration requires more rigorous and formalised methods, because the problems are more complex. For a complete use of operational research in a large organisation a master model will be constructed, co-ordinating the work of separate models, in each one of the specialised techniques described in this book.

Each chapter lays a simple foundation for a wide and important topic, and each bibliography contains key titles for further study. Readers will find that deeper penetration into specialised branches often depends upon more complex mathematical methods.

The authors are glad to acknowledge the great help received in planning and writing the book. Mr Shaie Selzer, one of the publisher's editors, was concerned with the birth of the book, and its early upbringing, and his successor, Mr Nicholas Brealey, saw it into publication. Mrs Sonia Yuan, B.A., M.Sc., Senior Lecturer in Statistics at Oxford Polytechnic, discussed with the authors the general proposals and saw some of the manuscript, and her advice has proved most valuable. Our three typists served in this taxing field most competently; they were Mrs Joan Jones and Miss Anne Westover, both of the Institute of Local Government Studies, Birmingham University, and Mrs Beryl Perry.

The authors have worked closely together to make the book useful, accurate and up to date, ideals in a developing quantitative field easier to set than to achieve; for any shortcomings they take full responsibility.

C. F. PALMER
A. E. INNES

Acknowledgements

The authors and publisher wish to thank the following, who have kindly given permission for the use of copyright material:

The Bank of England for an extract from *Bank of England Statistical Abstract*, no. 1, 1970.

The Controller of Her Majesty's Stationery Office for tables from *Monthly Digest of Statistics*, no. 375 (1975); *United Kingdom Energy Statistics* (1973); *Annual Abstract of Statistics* (1975), and *Housing and Construction Statistics*, no. 3.

The London and Cambridge Economic Service, University of Cambridge, for tables from *The British Economy Key Statistics 1900–1970*.

McGraw-Hill Book Company (U.K.) Ltd for a table from *Discounted Cash Flow*, 2nd ed. by M. G. Wright, ©1973.

Every effort has been made to trace all the copyright-holders, but if any have been inadvertently overlooked the publishers will be pleased to make the necessary arrangement at the first opportunity.

Contents

1

Introducing Operational Research

Operational research, or operations research (to use the American term), describes the application of specialised quantitative techniques to solving problems met in industry, in commerce and in administration. For many years separate statistical and mathematical methods had been used to guide decision-making in these fields, but it was the impetus of the Second World War in Great Britain and America that began to bring together teams of mathematicians, statisticians, psychologists, physicists and other scientists to tackle problems demanding over-all strategies of enquiry and application. Calculations of manpower and material needed to land and maintain combatant forces on hostile territory and of civilian food requirements in a siege economy were two examples of the fields in which operational research developed. Peaceful applications have been made in most countries of the world on lines described in the remaining chapters of this book.

Operational research studies *systems*, a term readily recognised but not easy to define. A fleet of lorries regularly delivering goods from a warehouse to a firm's customers constitutes a system. The term could properly be applied to surgeons working together in a hospital, with the anaesthetists, sisters and nurses working with them, and the equipment they use. In business, study of the behaviour of a system usually involves study of related sub-systems: the production-line of a factory is an identifiable system; but a change in its working would involve changes in the supplies of raw materials, in the employment of workers and perhaps in the storage of finished products. Essentially, therefore, a system is a group of people engaged in joint, purposeful activity, together with material means used to achieving it, within the general context of industry, commerce or administration; and such a system is likely to be supported by, or associated with, other systems called sub-systems.

Groups and sub-groups thus described rarely lend themselves to direct experiment of the kind carried out in laboratories in the natural sciences. A Birmingham firm, for example, with a new product to export may need to choose between Liverpool, Bristol and London as its outlet. It cannot set up three separate dock offices and in the light of experience choose the most suitable. A government may need to decide whether £100 million is better spent on electrifying the railway between two major cities or upon improving the road system between them. Not only are the issues at stake very complicated, but for the government there is little scope for crucial experiment, as the scientist would understand it. Both kind of problem are suitable for operational research, and insight would be obtained by setting up *models*.

Iconic models, i.e. those which give a physical resemblance to the original, are often used in engineering, architecture and other branches of study. The nautical properties of a proposed oil-tanker, for example, can be studied by constructing a small-scale model and simulating stormy weather in a laboratory. Operational research uses a variety of quantitative models and formulae which are mathematical or statistical in origin. A firm deciding upon its stock-ordering policy could use the simple model derived on page 90:

$$Q = \sqrt{\left(\frac{2DP}{SC}\right)}$$

where Q is the most economical size of batch in which to order stock, D is the total annual demand, P is the cost of placing one order, S is a measure of stock-holding costs and C is the unit cost of the items. This is an *a priori* model, a general description constructed from first principles. The expression

$$P = \frac{80}{Q + 65}$$

where Q is the number of units of a commodity demanded, and P is the unit price, is a simple model showing how prices change as demand changes, and which is likely to have been obtained by observing a number of pairs of values of P and Q for this commodity and fitting a relationship which best suits them. Such a model is *empirical* or *a posteriori*, though the general shape of the model, and in particular the positions of P and Q, reflect simple economic theory. The letters in both models indicate variables. Those which are determined by factors outside the defined system are called *exogenous*; for example, in the stock model, D, total demand, and the other factors on the right-hand side are determined by the size of the firm's market, salaries of order clerks, etc. Q, in this context, is an *endogenous* variable, because it influences the system from within. The batch size, for example, will affect the firm's storage and transport policies. In the second model Q will be exogenous and P endogenous.

The reason for constructing an operational research model is *optimisation*, i.e. the calculation of the best value for a particular set of conditions. In the stock model Q gives the size of batch which will give the lowest average figure of unit cost when the combined effect of purchase, ordering and holding are taken into account. An optimum may be a maximum value: we may, for example, in planning vehicle routes between a series of towns served by a delivery system choose that plan which makes the greatest use of vehicle capacity.

The two models quoted so far are comparatively simple. They are *static* in the sense that the only interrelationship between the factors involved is that described by the model; the effect does not itself influence the values of the factors causing it. But this is not always so. Consider the second model. In the long run a change in price will induce changes in quantities being produced for sale, and a more sophisticated model would be needed to do justice to the situation. Models which make allowance for changes induced in the system by its own operation are called *dynamic*. Where a model incorporates an allowance for the time taken for one factor to affect another, the variables are said to be *lagged*.

Where operational research methods are applied to large organisations the great problem is to reconcile the policies which models describing the separate parts would suggest. In a large manufacturing firm a model may show the scale of output which will minimise average production costs: but a model of the market may suggest a different figure for maximum profits. Neither of these levels may be consistent with the level of production which best suits the firm's capital structure and financial resources. Overall models which attempt to cover all the variables are sometimes constructed. Of necessity they are computer operated, and they constitute important tools of management. Operational research is scientifically based. Facts are studied objectively, hypotheses framed, tested and re-framed if necessary. Models which are constructed have the same logical status as theories in the natural sciences. Today they are indispensable tools of management. But the successful running of large enterprises, whether for public good or private profit, depends upon informed personal judgement, so that management becomes an art and not merely a science.

Bibliography

Most of the following books deal not only with the nature of operational research but also with the main topics covered by the other chapters of this book.

R. L. Ackoff and M. W. Sasieni, *Fundamentals of Operations Research* (New York: Wiley, 1968).

W. T. Bane, *Operational Research Models*, Occasional Paper No. 8 (London: H.M.S.O., 1968).

E. S. Buffa, *Operations Management* (New York: Wiley, 1976).
D. Gale, *Theory of Linear Economic Models* (New York: McGraw-Hill, 1960).
F. S. Hillier and G. T. Lieberman, *Introduction to Operations Research*, 2nd ed. (San Francisco: Holden–Day, 1974).
M. S. Makower and E. Williamson, *Operational Research – Problems, Techniques and Exercises*, 3rd ed. (London: Hodder & Stoughton, 1975).
G. H. Mitchell, *Operational Research: Techniques and Examples* (London: English Universities Press Ltd, 1972).
P. G. Moore, *Basic Operational Research*, 2nd ed. (London: Pitman, 1976).
F. Ricaferrera, *Operations Research Models for Business and Industry* (Cincinatti: South-Western Publishing Company, 1964).
P. Rivett, *Principles of Model Building: The Construction of Models for Decision Analysis* (New York: Wiley, 1972).
J. Singh, *Operations Research* (Harmondsworth: Penguin, 1971).

2

Queueing and Waiting-time Problems

An early application of operational research methods was to the problems of queueing. The queues formed in a modern supermarket by customers waiting to pay for baskets of goods typifies quantitative problems found in other business and industrial situations. Each cash-till is approached by a single *service-channel*, each customer is a *unit*, and the service-channels and tills or *service-points*, to use a more general term, form a *system*. The fundamental problem is to strike the right balance between customers' demands for services and the organisation's supply of service somewhere between the extremes of excessive queueing and uneconomical manning of service-points. The first example shows that even when the arrival rates of customers and the service times are fixed – a simple situation rarely met in practice – the system is very sensitive to small changes in rates or times.

Example 2.1 The stores of a large organisation issues material at the constant rate of 10 orders per hour. The stores open at 9.00 a.m. and workers arrive in succession at the rate of 8 per hour, as soon as the store opens. Assuming a single service-channel:

(i) For what proportion of the first hour will the storekeeper be issuing material?

(ii) What change in (a) arrival rate *or* (b) service rate would result in the storekeeper being fully employed without queueing occurring?

(iii) Investigate the queueing that would arise at the original service rate, but with workers arriving at a new rate of 12 per hour.

(i) An issue rate of 10 per hour means that one worker can be supplied in 6 min. But if workers arrive at 8 per hour, they will arrive at 60 min./8 = $7\frac{1}{2}$ mins., and there will be a gap of ($7\frac{1}{2}$ − 6) min. = $1\frac{1}{2}$ min. after each. Queueing will not occur, and the storekeeper will be employed

$$\frac{8 \times 6 \text{ min.}}{60 \text{ min.}} \times \frac{100}{1} = 80\% \text{ of the time}$$

(ii) Either (a) the storekeeper slows down to 8 per hour (the arrival rate), or (b) the workers increased their arrival rate to 10 per hour (the service rate).

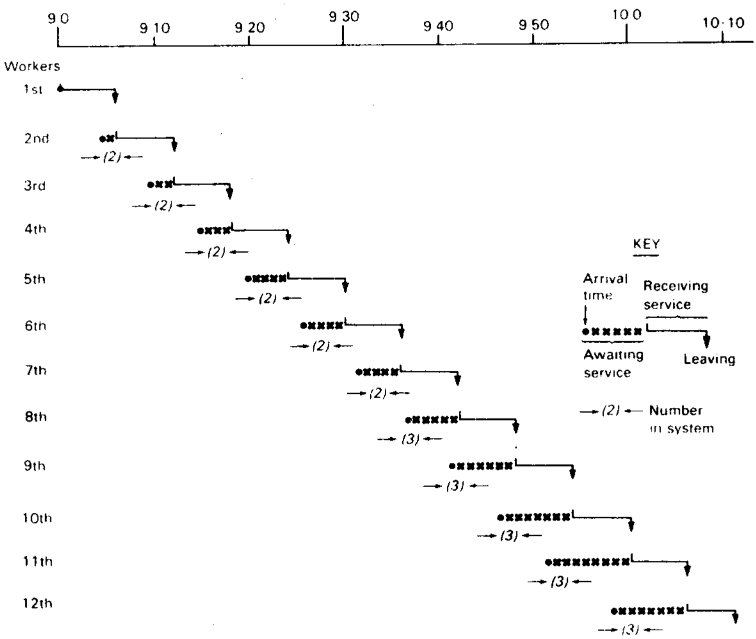


FIG. 2.1 Workers queueing at stores

(iii) Figure 2.1 shows how a queue would build up gradually. The second worker, arriving at 9.05 a.m., would queue for 1 min. before the storekeeper was free to start supplying him at 9.06 a.m. After 9.20 a.m. at least 2 people would be in the system. At 9.35 a.m. 3 would be in the system for a short time, and after 9.45 a.m. there would never be fewer than 3, until the arrival of workers stopped. Congestion would increase progressively and successive workers would have to wait increasingly longer. The twelfth worker, for example, would spend 16 min. in the system. With a single service point the number 'in the system' = 1 being served plus number waiting, so that from 9.45 a.m. the queue would contain 2 workers.

Under these simplified conditions whether queueing occurs depends upon the *traffic intensity*, a quantity indicated by ρ (Greek – pronounced rho), and calculated by dividing by the average number of units arriving in unit time λ (Greek lambda) divided by μ (Greek mu), the average number of services completed in unit time, i.e.

$$\rho = \frac{\lambda}{\mu}$$

In Example 2.1 the opening value of ρ equals

$$\frac{8 \text{ arrivals/hour}}{10 \text{ orders/hour}} = 0.8$$

for (ii) (a) $\frac{8}{8} = 1$ and (b) $\frac{10}{10} = 1$

and for (iii) $\frac{12}{8} = 1.5$

Where both μ and λ are unvarying the three cases just studied demonstrate that if $\rho < 1$ no queueing will occur and if $\rho = 1$ the service facility will be in continuous use, and if $\rho > 1$ queueing will start with the second arrival and will increase with subsequent arrivals.

In practice the above calculations have been oversimplified: no allowance has been made for the time that must elapse between finishing one service and starting the next. Initially this might appear trivial, but after several services a cumulative and disruptive lag would develop. An even more serious criticism can be made: neither the arrival rates nor the service times are likely to be rigid; variations in either will upset the neat calculations made so far and, where variations are combined, the disturbance to the original system will be great. In general if queueing is occurring, any increase in arrival rate or service time will make it worse, whereas a reduction in service time will only improve the situation if constantly matched with an increase in arrival rate.

If the two variables were completely random, queueing calculations would be almost impossible to make, though, given the limits of variation, methods of simulation (see Chapter 3) might be used. Fortunately, input and output in queueing situations can often be described by two statistical distributions, the first being introduced by the next example.

Example 2.2 Assume that the arrivals of workers in Example 2.1 follow the Poisson distribution, with $\lambda = 8$ per hour, and calculate the separate probabilities of 4, 5, 6, etc., up to 12 workers arriving in 1 hour.

The Poisson formula gives the probability of X events as follows:

$$P(x) = \frac{e^{-m} m^x}{X!},$$

when $X = 4$, $P(x) = \dfrac{e^{-8}8^4}{4!} = \dfrac{0.0003546 \times 4096}{4 \times 3 \times 2 \times 1} = 0.057$

and

when $X = 5$, $P(x)$ $\dfrac{e^{-8}8^5}{5!} = 0.092$

Other values, directly calculated, or more conveniently calculated by the recursion method (see *Business Statistics by Example*, by A. E. Innes (Macmillan, 1974, pp. 238–9), give

$$
\begin{aligned}
P(4) &= 0.057\\
P(5) &= 0.092\\
P(6) &= 0.122\\
P(7) &= 0.140\\
P(8) &= 0.140\\
P(9) &= 0.124\\
P(10) &= 0.099\\
P(11) &= 0.072\\
P(12) &= 0.048
\end{aligned}
$$

Probabilities decrease either side of the 4–12 range, which can be seen to account for 89.4 per cent of the probabilities. Hence, we can expect some divergency, but well over 50 per cent of the time arrivals are likely to be in the (6–10) per hour range. The use of formulae, to be stated presently, does not require Poisson calculations, but study of probabilities of the kind calculated above will show why queueing sometimes occurs well before ρ approaches unity.

The Poisson distribution calculates the frequency of events over set periods of time. Calculation of varying service times depends upon a distribution which is a kind of obverse of Poisson, because it deals with lengths of times between events. The next example introduces it.

Example 2.3 (i) Take the service times for Example 2.1 to be exponentially distributed, with expected frequency of 10 per hour. Calculate and graph the probabilities of service times of less than 6 min. and more than 6 min. (ii) Calculate by integration the proportion of service times that can be expected to be (a) between 5 min. and 7 min., and (b) between 3 min. and 9 min.

The probability density function for the exponential distribution is

$$Y = \mu e^{-\mu x}$$

where Y is the probability density, μ is the expected number of events in unit time and X the number of units of time that will elapse before the recurrence of an event, on the assumption that an event has just occurred.

Working in minutes, if 10 services occur in 1 hour, 1 occurs in 6 min., and in 1 unit of time, i.e. 1 min., 1/6 of a service will occur. Assuming no break occurs between the ending of a service and the beginning of the next, x is a service time. When $x = 0$ min.

$$Y = \frac{1}{6}e^{-1/6 \times 0} = \frac{1}{6} \times 1 = \frac{1}{6}$$

when $x = 1$ min.,

$$Y = \frac{1}{6}e^{-1/6 \times 1/1} = 0.1411$$

The following table summarises these and similarly calculated values:

Service time (min.)	*Probability density*
0	0.1667
1	0.1411
2	0.1194
4	0.0857
6	0.0613
8	0.0439
10	0.0315
20	0.0059
30	0.0011

(ii) (a) The required probability is given by the shaded area in Figure 2.2, and is obtained as the value of the probability density function integrated between the limits $x = 5$ and $x = 7$:

$$\int_5^7 \frac{1}{6}e^{\frac{-x/6}{\alpha x}} = \frac{1}{6}\left[1 - e^{-x/6}\right]_5^7 = \frac{1}{6}$$

$$\{(1 - 0.3114) - (1 - 0.4346)\} = 0.02053$$

(b) For the wider limits

$$\frac{1}{6}\left[1 - e^{-x/6}\right]_3^9 = 0.0639$$

Hence, the wide variations in service times are illustrated, only 2.053 per cent lying within a minute either side of the average, and only 6.39 per cent within 3 min. either side.

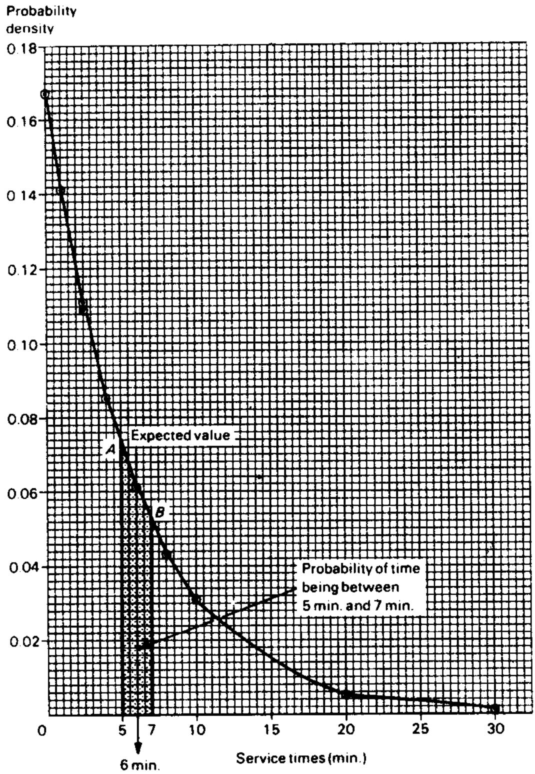


FIG. 2.2 Exponential distribution of service times

Such wide random variations in arrival and service times show why the approach of Example 2.1 needs considerable modification and why the measurements of queueing situations are usually in terms of probability. The combination of Poisson and exponential conditions has produced a series of formulae used to solve quantitative problems and the next examples introduce them.

Example 2.4 A tyre centre is open 10 hours per day for repairing punctures, the average repair time being 20 min. Customers arrive at an average rate of 20 per day. Calculate (i) the probability that a motorist has to wait upon arrival, and (ii) the number of hours during a 6-day working week when punctures are not being repaired.

(i) The probability, P, is the traffic intensity ρ already defined.

$\lambda = 20$ per 10-hour day $= 2$/hour
$\mu =$ average number of services in 1 hour $= 60$ min./20 min. $= 3$
Therefore $P = 2/3$ (= 66.7 per cent)

(ii) The probability of a motorist being 'in the system', i.e. either having a puncture repaired or waiting for it to be repaired, is given by ρ. Punctures will not be repaired when there is no one in the system, i.e.

the probability is $1 - P = 1 - 2/3 = 1/3$

in a full week for $\dfrac{6 \times 10}{3}$ hours = 20 hours.

Excessive queueing can be to the disadvantage of management not only because of physical congestion, but because customers may *renage*, i.e. leave the queue before being served. The next example introduces another formula used in such situations.

Example 2.5 A cashier at a supermarket check-out points deals with customers at an average rate of 30 per hour: (i) with customers arriving at an average rate of 25 per hour, calculate the average length of queue when a queue of 1 or more forms; (ii) what improvement is needed in service time if the average queue-length is to be reduced by 1?

(i) The formula for average length of queue is

$$\frac{1}{1 - \rho}$$

$\lambda = 25$/hour, $\mu = 60$ min./2 min. = 30/hour, giving

$$\rho = \frac{25}{30} = \frac{5}{6}$$

and average length is

$$\frac{1}{1 - 5/6} = 6$$

Note that the formula is based on the number of queues and does not reflect occasions when no queue occurs.

(ii) If the average is to be $(6 - 1) = 5$

$$\frac{1}{1-\rho} = 5$$

i.e. $1 = 5 - 5\rho$, giving $\rho = 0.8$

$$\frac{25}{\mu} = 0.8,$$

$$\text{i.e. } \mu = \frac{25}{0.8}/\text{hour},$$

giving an improved service time of

$$\frac{60}{25/0.8} \text{ min.} \simeq 1 \text{ min. } 55 \text{ sec.}$$

The next example introduces a formula which takes into account times when the queue length = 0 as well as occasions when it is greater than 0.

Example 2.6 (i) Use the data in Example 2.5(i) to calculate the average queue length on the basis just described. (ii) If the shop is open from 9.0 a.m. to 5.30 p.m., using the unimproved service rate, calculate the number of hours when no queue can be expected.

(i) The formula is

$$\frac{\rho^2}{1-\rho}$$

For $\rho = 5/6$, the average is

$$\frac{(5/6)^2}{1 - 5/6} = \frac{25}{6} = 4\tfrac{1}{6}$$

(ii) By the method of Example 2.4 (ii)

$$1 - \rho = 1 - 5/6 = 1/6$$

Total hours = $8\frac{1}{2}$, giving $8\frac{1}{2}/6$ hours = $1\frac{1}{4}$ hours without a queue.

Often we shall need to know the chance that a queue will be a particular length between nil and its maximum state. The probability of n people being in the system is given by

$$P(n) = (1 - \rho)\rho^n$$

and the next example uses it.