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Editors

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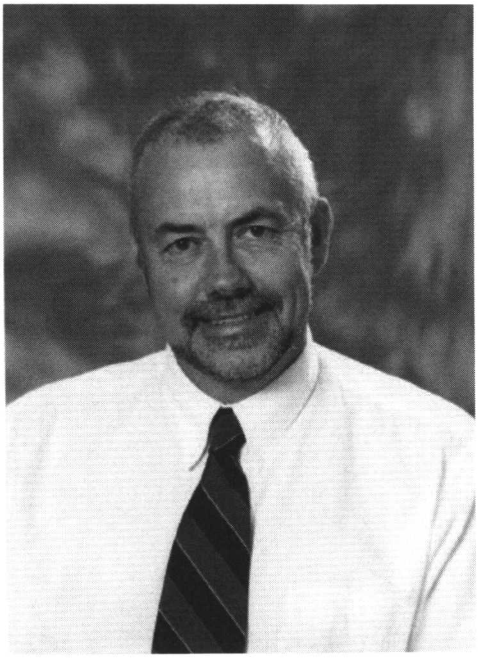

This volume is dedicated to the memory of Will Light who was a driving force in creating and running the first eight summer schools.

Preface

The Tenth LMS-EPSRC Numerical Analysis Summer School was held at the University of Durham, UK, from the 7th to the 19th of July 2002. This was the second of these schools to be held in Durham, having previously been hosted by the University of Lancaster and the University of Leicester. The purpose of the summer school was to present high quality instructional courses on topics at the forefront of numerical analysis research to postgraduate students. The speakers were Franco Brezzi, Gerd Dziuk, Nick Gould, Ernst Hairer, Tom Hou and Volker Mehrmann.

This volume presents written contributions from all six speakers which are more comprehensive versions of the high quality lecture notes which were distributed to participants during the meeting. At the time of writing it is now more than two years since we first contacted the guest speakers and during that period they have given significant portions of their time to making the summer school, and this volume, a success. We would like to thank all six of them for the care which they took in the preparation and delivery of their material.

Instrumental to the school were two groups: The five tutors who ran a very successful tutorial programme (Philip Davies, Sven Leyffer, Matthew Piggott, Giancarlo Sangalli and Vanessa Styles); the two "local experts", that is distinguished UK academics who, during the meeting, ran the academic programme on our behalf leaving us free to deal with administrative and domestic matters. These were Charlie Elliott (University of Sussex) and Sebastian Reich (Imperial College). In addition to chairing the main sessions the local experts also ran a successful programme of contributed talks from academics and students in the afternoons. The UKIE section of SIAM contributed prizes for the best talks given by graduate students. The local experts took on the bulk of the task of judging these talks. After careful and difficult consideration, and after canvassing opinion from other academics present, the prizes were awarded to Angela Mihai (Durham) and Craig Brand (Strathclyde). The general quality of the student presentations was impressively high promising a vibrant future for the subject.

The audience covered a broad spectrum, seventy-three participants ranging from research students to academics from within the UK and from abroad. A new feature of this meeting was that, thanks to the generosity of the LMS, we were able to fund a small number of students from continental Europe. As always, one of the most important aspects of the summer school was providing a forum for EU and UK numerical analysts, both young and old, to meet for an extended period and exchange ideas.

We would also like to thank the Durham postgraduates who together with those who had attended the previous Summer School ran the social

programme, Fionn Craig for dealing with registration, Rachel Duke, Tanya Ewart, Fiona Giblin, Vicky Howard and Mary Bell for their secretarial support and our families for supporting our efforts.

We thank the LMS and the Engineering and Physical Sciences Research Council for their financial support which covered all the costs of the main speakers, tutors, plus the accommodation costs of the participants.

James F. Blowey, Alan W. Craig and Tony Shardlow
Durham, March 2003

Contents

Subgrid Phenomena and Numerical Schemes

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Abstract. In recent times, several attempts have been made to recover some information from the subgrid scales and transfer them to the computational scales. Many stabilizing techniques can also be considered as part of this effort. We discuss here a framework in which some of these attempts can be set and analyzed.

1 Introduction

In the numerical simulation of a certain number of problems, there are physical effects that take place on a scale which is much smaller than the smallest one representable on the computational grid, but have a strong impact on the larger scales, and, therefore, cannot be neglected without jeopardizing the overall quality of the final solution.

In other cases, the discrete scheme lacks the necessary stability properties because it does not treat in a proper way the smallest scales allowed by the computational grid. As a consequence, some "smallest scale mode" appears as abnormally amplified in the final numerical results. Most types of numerical instabilities are produced in this way, such as the checkerboard pressure mode for nearly incompressible materials, or the fine-grid spurious oscillations in convection-dominated flows. See for instance [19] and the references therein for a classical overview of several types of these and other instabilities of this nature.

In the last decade it has become clear that several attempts to recover stability, in these cases, could be interpreted as a way of improving the simulation of the effects of the smallest scales on the larger ones. By doing that, the small scales can be *seen* by the numerical scheme and therefore be kept under control.

These two situations are quite different, in nature and scale. Nevertheless it is not unreasonable to hope that some techniques that have been developed for dealing with the latter class of phenomena might be adapted to deal with the former one. In this sense, one of the most promising technique seems to be the use of Residual-Free Bubbles (see e.g. [10], [18].) In the following sections, we are going to summarize the general idea behind it, trying to underline its potential and its limitations. In Section 2 we present the continuous problems in an abstract setting, and provide examples of applications, related to advection dominated flows, composite materials, and viscous incompressible flows. For application of these concepts to other problems we refer, for instance, to [13], [14], [16], [18], [24]. In Section 3 we introduce the basic features of the RFB method. Starting from a given discretization (that might possibly be unstable), we discuss the suitable *bubble space* that can be added to the original finite element space. Increasing the space with bubbles leads to the *augmented problem*, usually infinite dimensional, which, in the end, will have to be solved in some suitable approximate way. In Section 4 we give an idea of how error estimates can be deduced for the augmented problem. In Section 5 we discuss the related computational aspects, and we present several strategies that can be used to deal with the augmented problem, in order to minimize the computational cost. We shall see in particular that several other methods that are known in the literature can actually be seen as variants of the RFB procedure, in which one or another of the above strategies is employed. This includes, for advection dominated problems, the classical SUPG methods (as it was already well known, see, e.g., [4]) as well as the older Petrov-Galerkin methods based on suitable operator dependent choices of test and trial functions [25]. For composite materials, this includes both the multiscale methods of [22], [23], and the upscaling methods of [1], [2]. Finally, in Section 6 we draw some conclusions.

2 The Continuous Problem

We consider the following continuous problem

$$\begin{cases} \text{find } u \in V \text{ such that} \\ \mathcal{L}(u,v) = \langle f, v\rangle \qquad \forall\, v \in V, \end{cases} \tag{2.1}$$

where V is a Hilbert space, and V' its dual space, $\mathcal{L}(u,v)$ is a continuous bilinear form on $V \times V$, and $f \in V'$ is the forcing term. We assume that, for all $f \in V'$, problem (2.1) has a unique solution. Various problems arising from physical applications can be written in the variational form (2.1), according to different choices of the space V and the bilinear form $\mathcal{L}$. Typical choices for V, when V is a space of scalar functions, are the following: if $\mathcal{O} \subset \mathbb{R}^d$, $(d = 1,2,3)$ denotes a generic domain, V could be, for instance, $L^2(\mathcal{O})$, $H^1(\mathcal{O})$, $H^1_0(\mathcal{O})$, $H^2(\mathcal{O})$ or $L^2_0(\mathcal{O})$, the last one being the space of L^2-functions having zero mean value. In the case where V is a space of vector valued functions, a first choice could be to take the Cartesian product of the previous scalar spaces. Other typical choices for V can be:

$$H(\text{div};\mathcal{O}) := \{\boldsymbol{\tau} \in (L^2(\mathcal{O}))^d \text{ such that } \nabla\cdot\boldsymbol{\tau} \in L^2(\mathcal{O})\},$$
$$H_0(\text{div};\mathcal{O}) := \{\boldsymbol{\tau} \in H(\text{div};\mathcal{O}) \text{ such that } \boldsymbol{\tau}\cdot\mathbf{n} = 0 \text{ on } \partial\mathcal{O}\},$$

or also, for a generic domain $\mathcal{O} \subset \mathbb{R}^3$,

$$H(\mathbf{curl};\mathcal{O}) := \{\boldsymbol{\tau} \in (L^2(\mathcal{O}))^3 \text{ such that } \nabla\wedge\boldsymbol{\tau} \in (L^2(\mathcal{O}))^3\}$$
$$H_0(\mathbf{curl};\mathcal{O}) := \{\boldsymbol{\tau} \in H(\mathbf{curl};\mathcal{O}) \text{ such that } \boldsymbol{\tau}\wedge\mathbf{n} = 0 \text{ on } \partial\mathcal{O}\}.$$

Product spaces are also frequently used: for instance, $H(\mathrm{div};\mathcal{O}) \times L^2(\mathcal{O})$, or $(H_0^1(\mathcal{O}))^d \times L_0^2(\mathcal{O})$, etc. Next, we provide some classical examples of problems and we indicate the corresponding space V, the bilinear form $\mathcal{L}$, and the variational formulation.

Example 2.1. Advection-dominated scalar equations:

$$-\varepsilon\Delta u + \mathbf{c}\cdot\nabla u = f \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial\Omega$$
$$V := H_0^1(\Omega); \quad \mathcal{L}(u,v) := \int_\Omega \varepsilon\nabla u\cdot\nabla v\,dx + \int_\Omega \mathbf{c}\cdot\nabla u\, v\,dx; \quad \langle f,v\rangle := \int_\Omega fv\,dx$$
$$\mathcal{L}(u,v) = \langle f,v\rangle \qquad \forall\, v\in V.$$

Example 2.2. Linear elliptic problems with composite materials:

$$-\nabla\cdot(\alpha(x)\nabla u) = f \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial\Omega$$
$$V := H_0^1(\Omega); \quad \mathcal{L}(u,v) := \int_\Omega \alpha(x)\nabla u\cdot\nabla v\,dx; \qquad \langle f,v\rangle := \int_\Omega fv\,dx$$
$$\mathcal{L}(u,v) = \langle f,v\rangle \qquad \forall\, v\in V$$

(where $\alpha(x) \geq \alpha_0 > 0$ might have a very fine structure).

Example 2.3. Composite materials in mixed form, i.e., the same problem of the previous example, but now with:

$$\boldsymbol{\sigma} = -\alpha\nabla\psi \quad \text{in } \Omega; \qquad \nabla\cdot\boldsymbol{\sigma} = f \quad \text{in } \Omega; \qquad \psi = 0 \quad \text{on } \partial\Omega$$
$$V := \boldsymbol{\Sigma}\times\Phi; \qquad \boldsymbol{\Sigma} := H(\mathrm{div};\Omega); \qquad \Phi := L^2(\Omega)$$
$$a_0(\boldsymbol{\sigma},\boldsymbol{\tau}) := \int_\Omega \alpha^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx, \quad b(\boldsymbol{\tau},\varphi) := \int_\Omega \nabla\cdot\boldsymbol{\tau}\,\varphi\,dx$$
$$\mathcal{L}((\boldsymbol{\sigma},\psi),(\boldsymbol{\tau},\varphi)) := a_0(\boldsymbol{\sigma},\boldsymbol{\tau}) - b(\boldsymbol{\tau},\psi) + b(\boldsymbol{\sigma},\varphi); \quad \langle f,(\boldsymbol{\tau},\varphi)\rangle := \int_\Omega f\varphi\,dx$$
$$\mathcal{L}((\boldsymbol{\sigma},\psi),(\boldsymbol{\tau},\varphi)) = \langle f,(\boldsymbol{\tau},\varphi)\rangle \qquad \forall\,(\boldsymbol{\tau},\varphi)\in V.$$

Example 2.4. Stokes problem for viscous incompressible fluids:

$$-\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega; \qquad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega; \qquad \mathbf{u} = 0 \quad \text{on } \partial\Omega$$
$$V := \mathbf{U}\times Q; \quad \mathbf{U} := (H_0^1(\Omega))^d; \quad Q := L_0^2(\Omega)$$
$$a_1(\mathbf{u},\mathbf{v}) := \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\,dx \quad b(\mathbf{v},q) := \int_\Omega \nabla\cdot\mathbf{v}\,q\,dx$$
$$\mathcal{L}((\mathbf{u},p),(\mathbf{v},q)) := a_1(\mathbf{u},\mathbf{v}) - b(\mathbf{v},p) + b(\mathbf{u},q); \qquad \langle f,(\mathbf{v},q)\rangle := \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx$$
$$\mathcal{L}((\mathbf{u},p),(\mathbf{v},q)) = \langle f,(\mathbf{v},q)\rangle \qquad \forall\,(\mathbf{v},q)\in V.$$

3 From the Discrete Problem to the Augmented Problem

Let $\mathcal{T}_h$ be a decomposition of the computational domain Ω, with the usual nondegeneracy conditions [12], and let $V_h \subset V$ be a finite element space. The original discrete problem is then:

$$\begin{cases} \text{find } u_h \in V_h \text{ such that} \\ \mathcal{L}(u_h,v_h) = \langle f,v_h\rangle \qquad \forall\, v_h\in V_h. \end{cases} \tag{3.1}$$

Note that we do not assume that (3.1) has a unique solution. Indeed, the stabilization that we are going to introduce can, in some cases, take care of problems originally ill-posed. Our aim is, essentially, to solve eventually a final linear system having as many equations as the number of degrees of freedom of V_h. Apart from that, we are ready to pay some extra price, in order to have a better method. In some cases, the total amount of additional work will be small. In other cases, it can be huge. However, we want to be able to perform the extra work independently in each element so that we can do it, as a pre-processor, ***in parallel***. This implies that we are ready to add as many degrees of freedom as we want at the interior of each element. For that, to V and $\mathcal{T}_h$ we associate the *maximal space of bubbles*

$$B(V;\mathcal{T}_h) = \prod_K B_V(K), \qquad \text{with } B_V(K) = \{v \in V: \ \mathrm{supp}(v) \subseteq \overline{K}\}.$$

Let us give some examples of the dependence of $B_V(K)$ on V.

- if $V = H^1_0(\Omega)$ then $B_V(K) = H^1_0(K)$
- if $V = H^1(\Omega)$ then $B_V(K) = \{v \in H^1(K),\ v = 0 \ \mathit{on}\ \partial K \cap \Omega\}$
- if $V = L^2(\Omega)$ then $B_V(K) = L^2(K)$
- if $V = L^2_0(\Omega)$ then $B_V(K) = L^2_0(K)$
- if $V = H^2_0(\Omega)$ then $B_V(K) = H^2_0(K)$
- if $V = H_0(\mathrm{div};\Omega)$ then $B_V(K) = H_0(\mathrm{div};K)$
- if $V = H(\mathrm{div};\Omega)$ then $B_V(K) = \{\boldsymbol{\tau} \in H(\mathrm{div};K),\ \boldsymbol{\tau}\cdot\mathbf{n} = 0 \ \mathit{on}\ \partial K \cap \Omega\}$

Similar definitions and properties hold for the spaces $H(\mathbf{curl};\mathcal{O})$, but we are not going to use them here.

Let us now turn to the choice of the local bubble space $B_h(K)$. If possible, we would like to augment the space V_h by adding, in each element K, the whole $B_V(K)$. This would change V_h into $V_h + B(V;\mathcal{T}_h)$. However, some conditions are needed, as we shall see below. This might forbid, in some cases, taking the whole $B_V(K)$ in the augmentation process: some components of $B_V(K)$ have to be discarded. This will become more clear in the examples below. At this very abstract and general level, we assume that, in each $K \in \mathcal{T}_h$, we choose a subspace $B_h(K) \subseteq B_V(K)$ and, for the moment, "the bigger the better". A first condition that we require is that, for every $g \in V'$, the auxiliary problem

$$\begin{cases} \text{find } w_{B,K} \in B_h(K) \text{ such that} \\ \mathcal{L}(w_{B,K}, v) = \langle g, v\rangle \qquad \forall\, v \in B_h(K) \end{cases} \tag{3.2}$$

has a unique solution. We point out that the choice "the bigger the better" for $B_h(K)$ is made (so far) in order to understand the full potential of the method. As we shall see, in practice we will need to solve (3.2) a few times in each K. This implies that a finite dimensional choice for $B_h(K)$ will be, in the end, necessary.

Having chosen $B_h(K)$, we can now write the *augmented problem*. For that, let

$$V_A := V_h + \Pi_K B_h(K). \tag{3.3}$$

Two requirements have to be fulfilled: first of all, in (3.3) we must have a direct sum, and, second, for every $f \in V'$, the augmented problem

$$\begin{cases} \text{find } u_A \in V_A \text{ such that} \\ \mathcal{L}(u_A, v_A) = \langle f, v_A \rangle \qquad \forall\, v_A \in V_A \end{cases} \tag{3.4}$$

must have a unique solution. To summarize, in the augmentation process three conditions have to be fulfilled:

1) the local problems (3.2) must have a unique solution;
2) in (3.3) we must have a direct sum;
3) the augmented problem (3.4) must have a unique solution.

These are then the requirements that can guide us in choosing $B_h(K)$ in the various cases.

Examples of choices of $B_h(K)$.

Example 3.1. Referring to Examples 2.1 and 2.2 of the previous section, suppose that V_h is made of continuous piecewise linear functions. In this case it is easy to check that the choice $B_h(K) = B_V(K) \equiv H_0^1(K)$ verifies all of the three conditions.

Example 3.2. Suppose now that, always referring to Examples 2.1 and 2.2, V_h is made of continuous piecewise cubic functions. The choice $B_h(K) = B_V(K)$ is not viable anymore, as clearly condition 2) is violated: V_h contains functions of $B_V(K)$. In situations like this we should then choose a different $B_h(K)$, but we could also *reduce* the original space V_h. This is actually the simplest strategy, and we are going to follow it. Here, for instance, we can just remove the cubic bubble from $V_{h|K}$ and take a reduced space, still denoted by V_h with an abuse of notation, as a space of any serendipity cubic element (see, for instance, the element described in [12], page 50). Or we might take V_h as the space of functions v_h that are polynomials of degree ≤ 3 at the interelement boundaries and verify $Lv_h = 0$ separately in each K. Notice that these two choices produce the same augmented space V_A, and hence the same solution u_A to (3.4).

Example 3.3. Let us consider the problem of Example 2.3, and assume that $V_h = \Sigma_h \times U_h$ is made by lowest order Raviart-Thomas elements (see for instance [3]). For this problem we have

$$B_V(K) = \{\boldsymbol{\tau} \in H(\text{div}; K),\ \boldsymbol{\tau} \cdot \mathbf{n} = 0 \text{ on } \partial K \cap \Omega\} \times L^2(K).$$

we notice now that taking $B_h(K) = B_V(K)$ would not guarantee that problem (3.2) has a unique solution. Indeed, for internal elements K, the inf-sup condition is not satisfied, since $\int_K \operatorname{div}\tau\, v\, dx = 0\ \forall\ v$ constant on K. Condition 2) would also be violated by the choice $B_h(K) = B_V(K)$: in fact, U_h being the space of piecewise constants, $U_{h|K}$ contains bubbles of $L^2(K)$. A possible remedy in this case is to take

$$B_h(K) = H_0(\operatorname{div}; K) \times L^2_0(K) \subset B_V(K).$$

With this choice V_h remains the same, and B_h is the space of all pairs $(\tau, v) \in V$ such that τ has zero normal component at the boundary of each element, and v has zero mean value in each element. The same choice for B_h would be suitable also in the case of higher order Raviart-Thomas spaces (or, say, for BDM spaces; see always [3]), but then V_h should lose all internal degrees of freedom, apart from the piecewise constant scalars.

Example 3.4. Let us now examine the Stokes problem of Example 2.4, and assume that V_h is made of piecewise quadratic velocities in $(H^1_0(\Omega))^d$, and discontinuous piecewise linear pressures in $L^2_0(\Omega)$, a choice which is known not to be stable, but can be stabilized with the present technique. Actually, in this case one can see that $B_V(K) = (H^1_0(K))^d \times L^2_0(K)$. Taking $B_h(K) = B_V(K)$ would violate condition 2), but we can reduce the space V_h, taking it to be the space of quadratic velocities and *constant* pressures. It is easy to check that with this last choice we have a direct sum in (3.3). Moreover, problem (3.4) has a unique solution, because the inf-sup condition is now verified in V_A.

Example 3.5. Let us again consider the Stokes problem of Example 2.4, but now with $V_h = U_h \times Q_h$ made of piecewise linear continuous velocities in $(H^1_0(\Omega))^d$, and piecewise constant pressures in $L^2_0(\Omega)$. It is well known that for this choice the inf-sup condition does not hold. Moreover, if we augment V_h with bubble functions, in any way, the augmented problem (3.4) will **never** verify the inf-sup condition. To see that, augment the velocity space: $U_A = U_h + \Pi_K(H^1_0(K))^d$ as much as you can, and augment the pressure space: $Q_A = Q_h + \{0\}$ as little as you can. For every $v \in (H^1_0(K))^d$ and for every constant q in K, we clearly have $(\operatorname{div} v, q) = 0$. Hence, for $q \in Q_h$:

$$\sup_{v \in V_A} \frac{(\operatorname{div}\ v, q)}{\|v\|_1} = \sup_{v \in U_h} \frac{(\operatorname{div}\ v, q)}{\|v\|_1},$$

and we know that the last quantity cannot bound $\|q\|_0$ for all $q \in Q_h$. We clearly see that, in cases like this, our strategy is totally useless, and should not be applied.

4 An Example of Error Estimates

To give an idea of how to proceed to obtain error estimates, let us consider, as an example, a general singular perturbation problem where

$$\mathcal{L}(u, v) := \varepsilon a_1(u, v) + a_0(u, v)$$