V. A. Ilyin and E. G. Poznyak

Fundamentals of Mathematical Analysis

PART 2

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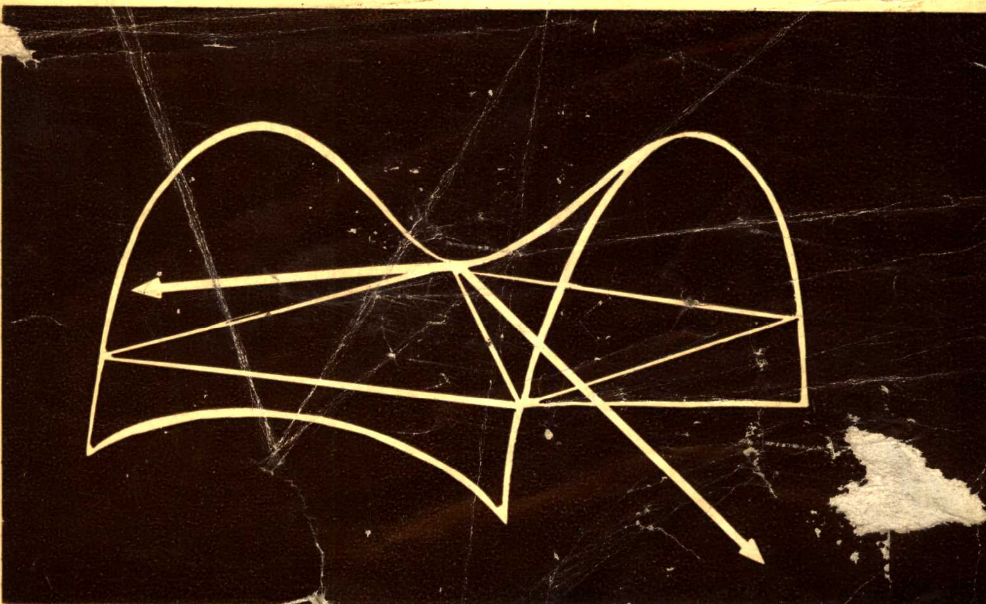

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ОСНОВЫ МАТЕМАТИЧЕСКОГО АНАЛИЗА

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ИЗДАТЕЛЬСТВО • МИР •

PREFACE

This book is based on the lectures read by authors at Moscow State University for a number of years.

As in Part 1, the authors strived to make presentation systematic and to set off the most important notions and theorems.

Besides the basic curriculum material, this book contains some additional questions that play an important part in various branches of modern mathematics and physics (the theory of measure and Lebesgue integrals, the theory of Hilbert spaces and of self-adjoint linear operators in these spaces, questions of regularization of Fourier series, the theory of differential forms in Euclidean spaces, etc.). Some of the topics, such as the conditions for termwise differentiation and termwise integration of functional sequences and functional series, the theorem on the change of variables in a multiple integral, Green's and Stokes's formulas, necessary conditions for a bounded function to be integrable in the sense of Riemann and in the sense of Lebesgue, are treated more generally and under weaker assumptions than usual.

As in Part 1, we discuss in this book some questions related to computational mathematics, including first of all approximate calculation of multiple integrals in the supplement to Chapter 2 and calculation of the values of functions from the approximate values of Fourier coefficients (A.N. Tichonoff's regularization method) in the Appendix.

The material of this book, together with that of Part 1 published earlier, constitutes an entire university course in mathematical analysis.

Note that throughout this text Part 1 is referred to as Volume 1 and designated [1]. It should also be stressed that when reading this book Chapter 8, The Lebesgue Integral and Measure, Chapter 11, Hilbert Space, and all the supplements may be skipped without impairing the understanding of the rest of the text.

The authors feel deeply indebted to A.N. Tichonoff and A.G. Sveshnikov for much valuable advice and numerous profound criticisms, to Sh.A. Alimov, who has done more than just editing this book, to L.D. Kudryavtsev and S.A. Lomov for a great number of valuable criticisms, to P.S. Modenov and Ya.M. Zhileikin, who have made available to the authors materials on field theory and approximate methods of evaluating multiple integrals.

V. Ilyin, E. Poznyak

CONTENTS

CHAPTER 1

FUNCTIONAL SEQUENCES AND FUNCTIONAL SERIES

In this chapter we shall study sequences and series whose members are not numbers but functions defined on some given set. Such sequences and series are widely used to represent the functions and to compute them approximately.

1.1. UNIFORM CONVERGENCE

1.1.1. The functional sequence and the functional series. *Let* $\{x\}$ *be some set*. Then, if we assign to each n of the natural numbers* $1, 2, \ldots, n, \ldots$ *by a definite rule some function* $f_n(x)$ *defined on* $\{x\}$, *the set of the numbered functions* $f_1(x), f_2(x), \ldots, f_n(x), \ldots$ *is said to be a* **functional sequence.**

The individual functions $f_n(x)$ are called *members* or *elements* of the sequence, and $\{x\}$ is its *domain of definition* or simply domain.

The symbol $\{f_n(x)\}$ will be used to designate a functional sequence.

The formally written sum

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots \qquad (1.1)$$

of an infinite number of elements of a functional sequence $\{u_n(x)\}$ will be called a *functional series.*

The terms $u_n(x)$ of that series are functions defined on some set $\{x\}$.

The set $\{x\}$ is called the *domain of definition*, or domain, of the functional series (1.1).

As in the case of the number series, the sum of the first n terms of (1.1) are called the nth *partial sum* of that series.

It should be stressed that the *study of functional series is perfectly equivalent to the study of functional sequences*, for to every functional series (1.1) uniquely corresponds a functional sequence

$$S_1(x), S_2(x), \ldots, S_n(x), \ldots \qquad (1.2)$$

* In particular, by $\{x\}$ we may imply both the set of points of a straight line and the set of points $x = (x_1, x_2, \ldots, x_m)$ of a Euclidean space E^m.

of its partial sums, and, conversely, to every functional sequence (1.2) uniquely corresponds a functional series (1.1) with terms

$$u_1(x) = S_1(x),\ u_n(x) = S_n(x) - S_{n-1}(x) \quad \text{for} \quad n \geqslant 2$$

for which the sequence (1.2) is a sequence of partial sums.

Here are examples of functional sequences and series.

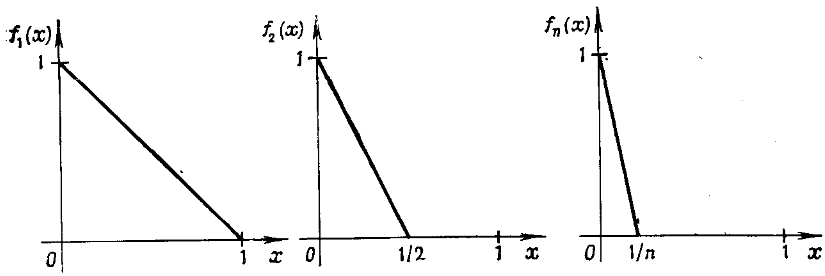

Fig. 1.1

Example 1. Consider a sequence of functions $\{f_n(x)\}$ each defined on the closed interval $0 \leqslant x \leqslant 1$ and having the form

$$f_n(x) = \begin{cases} (1 - nx) & \text{when } 0 \leqslant x \leqslant 1/n, \\ 0 & \text{when } 1/n \leqslant x \leqslant 1. \end{cases} \tag{1.3}$$

Figure 1.1 gives the graphs of the functions $f_1(x)$, $f_2(x)$ and $f_n(x)$.

Example 2. As an example of a functional series consider the following power series in x:

$$1 + \sum_{k=1}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \tag{1.4}$$

Notice that the $(n+1)$th partial sum of (1.4) differs from the Maclaurin expansion of e^x only by the remainder term $R_{n+1}(x)$.

1.1.2. Convergence of a functional sequence at a point and on a set. Suppose a functional sequence (or series) is defined on a set $\{x\}$. Fix an arbitrary point x_0 of $\{x\}$ and consider all the elements or the sequence (or the terms of the series) at x_0. We obtain a number sequence (or series).

If this number sequence (or series) converges, the given functional sequence (or series) is said *to converge at* x_0.

The set of all points x_0 at which a given functional sequence (or series) converges is called the *domain of convergence* of that sequence (or that series).

At various particular cases the domain of convergence may either coincide with the domain of definition or form a part of the domain or be an empty set.