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Message from the General Chair

On behalf of Enric Trillas and myself, I wish to welcome you to Santiago de Compostela, the site of the 26th International Symposium for Multiple-Valued Logic. This symposium is sponsored by the University of Santiago de Compostela, the Commission for the Fifth Centennial Anniversary of the University of Santiago de Compostela, and by the IEEE Computer Society. The conference would like to gratefully acknowledge the generous support given by the General Director of Scientific and Technical Research (Ministry of Education and Science), the Honorable Council of Santiago de Compostela, and the Government of Galicia.

We would also like to recognize all of the special efforts made by the local organizing committee with special thanks to Alejandro Sobrino, Senen Barro, and Alberto Bugarin, whose dedicated efforts have definitely contributed to the overall success of this symposium.

ISMVL'96 would especially like to honor the memory of a great Polish mathematician, Professor Helena Rasiowa. As in the past, our symposium has attracted researchers from a wide variety of disciplines including mathematicians, logicians, engineers, and computer scientists, helping to continue making this a dynamic and exciting gathering. The credit for putting together this exciting and thought-provoking program goes to the organizing and program committees. The following people graciously served as program chairs for their respective regions: Charles Silio for the Americas, Claudio Moraga for Europe and Africa, and Tsutomu Sasao for Asia and Australia.

We also wish to thank Regina Spencer Sipple of the IEEE Computer Society Press for her diligent work in publishing this volume.

Dan A. Simovici
ISMVL-96 Symposium Co-Chair

Message from the Program Chair

Charles B. Silio, program co-chair for America, Tsutomu Sasao, program co-chair for Asia and the Pacific, and I welcome you to the 1996 International Symposium on Multiple-Valued Logic in the frame of the celebrations of the 5th Centennial of the University of Santiago de Compostela, Spain.

The technical program of selected contributed papers consists of 46 papers by 92 authors currently working in 17 different countries. The ISMVL '96 Program Committee gratefully acknowledges the important support given by one hundred of referees from 14 countries. These referees prepared written reviews of each submitted paper and had to work under extremely tight time schedules.

Sessions of contributed papers deal with exciting innovations and research in Algebra, Logic, Switching Theory, Devices, Artificial Intelligence, Fault Modeling and Diagnosis, Soft Computing, Logic Design, and Decision Diagrams. This breadth of related topics reflects the diverse interests of the mathematicians, logicians, engineers, philosophers, and computer scientists who have come together to share their common interest in multiple-valued logic and to participate in this symposium.

In addition to the 46 contributed papers, we are pleased to present lectures by two renowned keynote speakers: Claudi Alsina, of the Open University of Catalunya, will discuss *Connectives in Fuzzy Logic*, and Lotfi Zadeh, of the University of California at Berkeley, will lecture on *Inference in Fuzzy Logic via Generalized Constraint Propagation*.

Moreover, a special session has been dedicated to honor the memory of the late Helena Rasiowa. The invited speakers for this session are G. Malinowski, Lódz University, Poland; J.M. Font, the University of Barcelona, Spain; and Tom Sales, Universitat Politècnica de Catalunya, Spain.

We sincerely hope that you enjoy the thought-provoking program we have put together and that you will find the motivation to return in the future for more research and camaraderie.

Welcome to ISMVL '96!

Claudio Moraga
Program Chair

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Keynote Address I

As You Like Them: Connectives in Fuzzy Logic

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Universitat Oberta de Catalunya

As You Like Them: Connectives in Fuzzy Logic

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Abstract

We review the question of which connectives (conjunctions, disjunctions and negations) may be of interest in Fuzzy Logic. Several alternative structures to the classical boolean algebras are presented and discussed. We show how techniques from the theory of functional equations may help to clarify the problem of choosing appropriate connectives or refusing inadequate operations.

1 Introduction.

In this work we would like to review a very basic problem of Fuzzy Logic: which conjunctions, disjunctions and negations may play a crucial role. While in classical set theory it is obvious the importance of boolean algebras, in the context of fuzzy sets, several alternative structures may be considered.

In the last thirty years a lot of literature in Fuzzy Logic has been devoted to present various approaches to the problem of determining logical connectives. These works have benefited from results arising in the theory of functional equations and in the field of probabilistic metric spaces. So today's problem is, mainly, to choose which families of connectives may be of interest and to clarify why some elections make sense and others do not merit consideration. Our aim here is to present a short survey of what has been reached and what requires further analysis.

2 Fuzzy Sets, Functional Equations and Probabilistic Metrics

Since 1965, when Lotfi Zadeh ([24]) founded the theory of Fuzzy Sets, there has been an explosion of interest both in the mathematical aspects of the theory and in the practical impact of it. Fuzzy Sets theory has been using a good deal of classical mathematical notions but, what is more important, the theory has motivated the development of interesting new mathematical machineries and results. The case of connectives in Fuzzy Logic is a clear example of a problem which has been a stimulating focus for mathematical research.

The field of Functional Equations goes back to antiquity, if one considers some old geometrical definitions of curves, but began its real development almost simultaneously with the appearence of the modern concept of function. Nevertheless the basic foundation has been made by J nos Acz l, whose celebrated book [2], widely known after 1966, has become the very basic reference in the field (see also [3]). Functional equations are tools for modelling a wide variety of practical problems and may be used for solving many questions formulated in terms of functional relations. Let us mention here that functional equations may be used to define classes of membership functions, fuzzy relations, fuzzy equations, etc. As we will see later the results on the associativity equation have become a basic tool for the study of connectives.

In 1942, Karl Menger (see [15]) introduced the pioneer ideas of probabilistic metric spaces. Berthold Schweizer and Abe Sklar began, after 1960, to develop this theory but found immediately the need to work with semigroups in real intervals and in the space of probability distribution functions. These problems motivated Schweizer and Sklar to deal with special classes of solutions of the associativity equation. Many other people, following results of Schweizer and Sklar, became later interested in the semigroups called t-norms and used them as generalized logical connectives.

Thus it is interesting to note that in the 60's three different fields like Fuzzy Logic, Probabilistic Metrics and Functional Equations benefited each other from problems related to fuzzy structures. Since those days the relations have shown to be an interesting source of new mathematical results.

3 Some remarks on a class of associative functions

The functional equation of associativity was first considered by Abel in 1826. This work motivated a question included by David Hilbert in his celebrated 1900 address. Between 1909 and 1948, several representation theorems were found by L.E. Brouwer and E. Cartan and between 1955 and 1963 several results were given, in the context of topological semigroups, by A.D. Wallade, Faucett, Mostert and Shileds, Clifford, Fuchs, etc. But the fundamental representation theorem for associative functions was proved by Acz l in 1949 (see [1], [2], [3]) and a basic extension was given by Ling in [13]. The literature devoted to various generalizations and extensions to the original cases of Acz l and Ling is quite impressive ([4]).

The study of the triangle inequality for probabilistic

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metrics induced to Schweizer and Sklar to study the following concepts:

Definition 3.1 *A* **t-norm** *is a two-place function T from $[0,1] \times [0,1]$ into $[0,1]$ such that the following conditions are satisfied for all x, x', y, y' and z in [0,1]:*

(i) Associativity: $T(x, T(y,z)) = T(T(x,y), z)$;

(ii) Commutativity: $T(x,y) = T(y,x)$;

(iii) Monotonicity: $T(x,y) \leq T(x',y')$ *whenever* $x \leq x'$ *and* $y \leq y'$;

(iv) Unit element: $T(x,1) = T(1,x) = x$;

(v) Null element: $T(x,0) = T(0,x) = 0$.

The most celebrated t-norms are

$$\begin{aligned} \text{Min}\,(x,y) &= \text{Minimum}\,\{x,y\}, \\ \text{Prod}\,(x,y) &= x \cdot y, \\ W(x,y) &= \text{Max}\,(x+y-1, 0) \end{aligned}$$

Definition 3.2 *A* **strict involution** *or* **strong negation** *on [0,1] is a function N from [0,1] onto [0,1] which is strictly decreasing, $N(0) = 1$, $N(1) = 0$ and $N \circ N = j$.*

The classical strong negation is $1 - j$, i.e., $(1-j)(x) = 1 - x$.

Definition 3.3 *A* **t-conorm** *is a binary operation S on [0,1] such that $S^*(x,y) = 1 - S(1-x, 1-y)$ is a t-norm.*

Let us quote a fundamental representation theorem for t-norms in its latest version:

Theorem 3.1 *Let T be a two-place function from $[0,1]^2$ into $[0,1]$ such that:*

(i) $T(x,0) = T(0,x) = 0$,

(ii) $T(1,1) = 1$,

(iii) T is associative,

(iv) T is jointly continuous.

Then T admits one of the following representations:

(a) $T(x,y) = Min\,(x,y)$;

(b) $T(x,y) = t^{(-1)}(t(x) + t(y))$, *where t is a continuous and strictly decreasing function from $[0,1]$ into R^+, with $t(1) = 0$ and $t^{(-1)}$ is the pseudo-inverse of t;*

(c) There exists a countable collection $([a_n, b_n])$ of non-overlapping, closed, non-degenerate subintervals of $[0,1]$ and a collection of t-norms T_n) each of them representable in the form (b) such that

$$T(x,y) = \begin{cases} a_n + (b_n - a_n) T_n\left(\frac{x-a_n}{b_n-a_n}, \frac{y-a_n}{b_n-a_n}\right), \\ \qquad \text{if } (x,y) \text{ in } [a_n,b_n]^2 \text{ for some } n, \\ Min\,(x,y), \quad \text{otherwise.} \end{cases}$$

This theorem shows that there is a wonderful collection of t-norms which have interesting representations for computational purposes and that, indeed, if the operation is jointly continuous there is no need to require neither monotonicity nor commutativity.

The previous theorem yields a corresponding representation for all continuous t-conorms. A representation for strong negations in the form $N(x) = f^{-1}(1 - f(x))$ was given by Enric Trillas ([18]).

4 Connectives in Fuzzy Logic

Given a nonempty base set X of elements of interest we can consider the boolean algebra $(P(X), \cap, \cup, \mathcal{C})$. This structure has a correspondence with the classical logic approach to disjunctions, conjunctions and negations of propositions and, by identifying subsets of X with their characteristic functions, we may say that the boolean operations on $P(X)$ are based upon the trivial boolean algebra ($\{0,1\}$, Min, Max, $1-j$).

Since fuzzy subsets of X will be functions μ from X into $[0,1]$ and we would like to consider, in a very general framework, pointwise operations, we find immediately the need to face the problem of which binary operations T and S in $[0,1]$ and which mappings N from [0,1] into [0,1] are such that the structure $([0,1], T, S, N)$ is "satisfactory" or "convenient". If it is possible to find a good answer (or several!) to this problem, then the basic structures in Fuzzy Logic would be clarified.

The initial proposal of Zadeh for the structure (T,S,N) was to consider (Min, Max, $1-j$), i.e., to perform the intersection, union and complements of fuzzy sets by means of the operation

$$\begin{aligned} (\mu_A \wedge \mu_B)(x) &= \text{Min}\,(\mu_A(x), \mu_B(x)), \\ (\mu_A \vee \mu_B)(x) &= \text{Max}\,(\mu_A(x), \mu_B(x)), \\ \mu'_A(x) &= 1 - \mu_A(x). \end{aligned}$$

This is a convenient extension of the classical case but some of the boolean properties are lost, e.g., $\mu_A \vee \mu'_A \neq 1$ and $\mu_A \wedge \mu'_A \neq 0$.

It was quite clear from the very beginning that for fuzzy sets it would be necessary to consider structures less restrictive than the classical boolean case. An easy analysis of the situation showed that strong conditions like idempotency ($\mu_A \wedge \mu_A = \mu_A \vee \mu_A = \mu_A$) or distributivity would restrict the possible operations to the couple (Min, Max), which did not satisfy, in [0,1], relations obviously true in $\{0,1\}$. Since (Min, Max, $1-j$) was a particular case of a triple (T,S,N) with T a t-norm, S a t-conorm and N a strong negation then several authors and, in special, Trillas' school, began to study the triples (T, S, N). Since the representation theorem for t-norms was an essential tool, all properties of continuous t-norms were considered, even the non-decreasing character of T, which in origin was required in order to deal with probability distribution functions and in the fuzzy context was assumed under the idea that t-norms would respect the usual pointwise ordering of fuzzy sets. In that moment the representation theorem without monotonic assumptions was not known.

5 On some classes of basics triples

We begin with the following:

Definition 5.1 *A* **basic triple** (T, S, N) *is given by two binary operations* T *and* S *on* $[0,1]$ *and a mapping* $N : [0,1] \to [0,1]$ *such that the following nine properties are satisfied, for all* x, y *and* z *in [0,1]:*

(1) $T(x, T(y, z)) = T(T(x, y), z)$;

(2) $T(x, y) = T(y, x)$;

(3) $T(x, 1) = x$;

(4) $T(x, 0) = 0$;

(5) $S(x, S(y, z)) = S(S(x, y), z)$;

(6) $S(x, y) = S(y, x)$;

(7) $S(x, 1) = 1$;

(8) $S(x, 0) = x$;

(9) $N(N(x)) = x$, $N(1) = 0$, $N(0) = 1$.

Thus a t-norm T, a t-conorm S and a strict involution N constitute a basic triple (T, S, N). When T and S are jointly continuous as two-place functions and N is continuous we have for (T, S, N) the representations given above.

But let us note that if no continuity is involved or no monotonicity is required then we can have basic triples which can not be represented in an easy way.

Example 5.1 Consider the triple $(T, T^*, 1-j)$ where T is the binary operation defined by

$$T(x,y) = \begin{cases} x, & \text{if } y = 1, \\ y, & \text{if } x = 1, \\ \text{Min } (x,y), & (x,y) \in \Big[[0,1] \cap Q\Big]^2 \cup \Big[[0,1] \setminus [0,1] \cap Q\Big]^2, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have a basic triple such that $T(x, x) = x$ for all x in [0,1] but T is nowhere monotonic on $(0,1)^2$.

Example 5.2 Consider $(T, T^*, 1-j)$ where T is given by

$$T(x,y) = \begin{cases} \text{Min } (x,y), & x = 1 \text{ or } y = 1, \\ xy/4, & 0 \le x,\ y \le 1/2,, \\ xy/2, & 1/2 < x,\ y \le 1, \\ xy/2\sqrt{2}, & \text{otherwise.} \end{cases}$$

Then T is a discontinuous strictly increasing Archimedean t-norm with discontinuity points on the interior points of the unit square.

Let us consider now basic triples which satisfy some additional conditions.

Definition 5.2 *A* **De Morgan triple** *is a basic triple* (T, S, N) *such that* S *and* T *are* N*-dual, i.e., we have the additional property*

(10) $N(T(x, y)) = S(N(x), N(y))$ *or*

(10') $N(S(x, y)) = T(N(x), N(y))$.

The study of N-duality was made by Garcia and Valverde in the case of continuous t-norms and t-conorms.

Definition 5.3 *A* **Lukasiewicz triple** *is a De Morgan triple such that the following condition holds*

(11) $T(x, y) = 0$ *if and only if* $y \le N(x)$.

Let us note that condition (11) and the assumed N*-duality of* S *with respect to* T*, yield that (11) is equivalent to*

(11') $S(x, y) = 1$ *if and only if* $N(y) \le x$.

It is interesting to note that Lukasiewicz triples are the natural solutions to the orthomodularity property

$$T(x, S(y, n(x))) = y, \text{ whenever } y \le x,$$

or to the strict local modularity:

$$T(x, S(y, z)) = S(y, T(x, z)),$$

whenever $y \le x$ and $N(x) \le z \le N(y)$.

Definition 5.4 *A basic triple* (T, S, N) *will be said* **idempotent** *if for all* x *in [0,1] we have*

(12) $T(x, x) = x$;

and

(13) $S(x, x) = x$.

If T is a t-norm then (12) yields $T = \text{Min}$ and if S is a t-conorm from (13) we deduce $S = \text{Max}$. Thus while idempotency is a natural rich property for some classes of operations in [0,1] like the averaging functions (means), it is quite unnatural for associative functions with boundary conditions related to the end points of [0,1] and with some monotonicity. We may remember here that even George Boole gave special arguments to include this property in his "algebraic" model.

But if we have basic triples with no monotonicity required then we may find some bizarre solutions.

Example 5.3 Let T be a binary operation in [0,1] defined by

$$T(x,y)=\sum_{n=1}^{\infty} x_n y_n/2^n,$$

where $x=\sum_{n=1}^{\infty} x_n/2^n$, $y=\sum_{n=1}^{\infty} y_n/2^n$ are well-defined dyadic expansions of x, y with $x_i, y_i \in \{0,1\}$. Consider $S(x,y)=T^*(x,y)$. Then $(T,S,1-j)$ is an idempotent basic triple.

Note that between Min and Max we may find also interesting associative, non-decreasing, continuous, commutative binary operations which satisfy (12) or (13) but the boundary conditions of a basic triple cannot be assumed. One example is to define, for any c in (0,1):

$$T_c(x,y)=\begin{cases} \text{Min }(x,y), & \text{if } x,y\le c,\\ \text{Max }(x,y), & \text{if } x,y\ge c,\\ c, & \text{otherwise,}\end{cases}$$

and consider $(T_c, T_c^*, 1-j)$.

Definition 5.5 *A basic triple (T,S,N) will be said* **distributive** *if it satisfies the following two conditions for all x, y and z in [0,1]:*

(14) $T(x,S(y,z))=S(T(x,y),T(x,z))$,

and

(15) $S(x,T(y,z))=T(S(x,y),S(x,z))$.

Note that (14) yields $S(x,x)=x$ and (15) implies $T(x,x)=x$. Thus if T is a t-norm and S is a t-conorm, only (Min, Max, N) is a distributive triple. But in the case of a basic triple with no monotonicity we may find other solutions, e.g., the operations given above in Example 5.3 and those given later in Example 6.1.

Note that basic triples have been used in Fuzzy Logic in various situations ([5], [12], [14], [19], [20]):

(a) As logical connectives to be used for making conjunctions, disjunctions and negations;

(b) To define implication functions, e.g.,

$$\begin{aligned} I(x,y) &= S(n(x),y)\\ R_T(x,y) &= \sup\{z \text{ in } [0,1] \mid T(z,x)\le y\}\\ Q(x,y) &= S(n(x),T(x,y))\end{aligned}$$

(c) To model some general "rules", e.g., the modus ponens inequality

$$T(x,I(x,y))\le y$$

(d) To define special properties of fuzzy relations, e.g.,

$$T(E(x,y),E(y,z))\le E(x,z),$$

which corresponds to the T-transitivity of a relation E in [0,1].

6 On Frank's triples

In 1979, M.J. Frank proved a remarkable result which has a lot of implications for the theory of semigroups on an interval, for the study of operations in the space of distribution functions and, as we will see, for the foundations of fuzzy logic operations. Frank's result concerns the study of which continuous t-norms T and t-conorms S may satisfy the functional equation

$$T(x,y)+S(x,y)=x+y. \qquad (*)$$

Theorem 6.1 *A continuous t-norm T and a t-conorm S satisfy equation (*) if and only if the couple (T,S) has one of the following forms:*

(i) $T_0(x,y)=Min\ (x,y)$, $S_0(x,y)=Max\ (x,y)$;

(ii) $T_1(x,y)=Prod\ (x,y)$, $S_1(x,y)=Prod^*(x,y)$;

(iii) $T_\infty(x,y)=W(x,y)$; $S_\infty(x,y)=W^*(x,y)$;

(iv) $T_S(x,y)=\log_S[1+(S^x-1)(S^y-1)/(S-1)]$, $0<S<\infty$, $S\ne 1$, $S_S(x,y)=T_S^*(x,y)$;

(v) T is representable as an ordinal sum of t-norms each of which is a member of the family $T_S(0<S\le\infty)$, and $S(x,y)=x+y-T(x,y)$.

It is interesting to note the following facts concerning the solutions of equation (*):

(a) All solutions are operations between W and Min which are, indeed, copulas;

(b) All solutions in the family T_S, with $0<S<\infty$, are smooth and have convenient differential properties;

(c) As a consequence of the equation (*) all Archimedean t-conorms $S_s(x,y)$, $0<s\le\infty$, given by $x+y-T_S(x,y)$ are at the same time the $(1-j)$-duals of T_s, i.e., T_S satisfy the equation:

$$x+y-T_s(x,y)=1-T_s(1-x,1-y).$$

(d) Equation (*) can be presented in the form

$$T(x,y)+S(x,y)=x+y=\text{Max }(x,y)+\text{Min }(x,y),$$

whence

$$\text{Min }(x,y)-T(x,y)=S(x,y)-\text{Max }(x,y),$$

and this is an important property to be required.

Thus we will consider from now on the following

Definition 6.1 *A basic triple (T,S,N) will be said to be a* **Frank's triple** *if it satisfies equation (*).*

We have seen above **all** Frank's triples determined by continuous t-norms and t-conorms. The following example shows that very sophisticated Frank's triples may be constructed without continuity or monotonicity properties.

Example 6.1 Let (λ_n) be an interval filling sequence in [0,1], i.e., $\lambda_n > \lambda_{n+1}$ for all n, $\sum_{n=1}^{\infty} \lambda_n = 1$ and for any x in [0,1] there exists a unique factorization of x in the form $x = \sum_{n=1}^{\infty} x_n \lambda_n$ with $x_n \in \{0,1\}$, for all n. Let us define T_λ and S_λ as binary operations in [0,1] given, respectively by,

$$T_\lambda(x,y) : = \sum_{n=1}^{\infty} \text{Min}\,(x_n, y_n)\lambda_n,$$
$$S_\lambda(x,y) : = \sum_{n=1}^{\infty} \text{Max}\,(x_n, y_n)\lambda_n,$$

whenever $x = \sum_{n=1}^{\infty} x_n\lambda_n$ and $y = \sum_{n=1}^{\infty} y_n\lambda_n$.

Then T_λ and S_λ are non-monotonic operations such that $([0,1], T_\lambda, S_\lambda)$ is a distributive lattice, $T_\lambda(x,y) + S_\lambda(x,y) = x + y$ and S_λ is not an N-dual operation of T_λ for all negations N.

7 On non-dual basic triples

In some cases it may be natural to deal with basic triples (T,S,N) where some relations link T, S and N but no direct duality between T and S is possible.

Definition 7.1 *A basic triple (T,S,N) will be called a* **normal triple** *if the following law holds for all x, y in [0,1]:*

$$S\,(T(x,y), T(x,N(y))) = x.$$

In the case where T is a continuous t-norm, S is a continuous t-conorm and N is a continuous strict involution it has been proved by this author that the three unknown functions can be represented in the form

$$\begin{aligned} S(x,y) &= s^{(-1)}(s(x) + s(y)), \\ T(x,y) &= s^{(-1)}(s(x) \cdot s(y)), \\ N(x) &= s^{(-1)}(1 - s(x)), \end{aligned}$$

where $s : [0,1] \to [0,1]$ is a continuous increasing additive generator for S, with $s(1) = 1$, $s(0) = 0$.

It's interesting to note that the solutions obtained are not N-duals and, indeed, since T is strict and S is a non-strict Archimedean t-conorm, T and S cannot be n-duals for all strong negations n.

Alsina and Trillas ([5]) have recently proved the following result related to the study of conditional and implication functions

Theorem 7.1 *Let T be a non-strict Archimedean t-norm, i.e., $T(x,y) = t^{(-1)}(t(x) + t(y))$ with $t(0) = 1$. Let $N(x) = t^{-1}(1 - t(x))$ and let S be a continuous t-conorm. Define*

$$T_1(x,y) := T\,(x, S(N(x), y))\,,$$

Then T_1 is a continuous t-conorm if and only if:

(i) $T_1 = T$, $S = Max$;

(ii) $T_1 = Min$, $S(x,y) = N(T(N(x), N(y)))$;

(iii) $T_1(x,y) = t^{-1}\,(1 - (1 - t(x))(1 - t(y)))$, $S(x,y) = t^{-1}(t(x) \cdot t(y))$;

(iv) $T_1(x,y) = t^{-1}\left(T^*_{1/\alpha}\,(t(x), t(y))\right)$, $S(x,y) = t^{-1}\,(T_\alpha\,(t(x), t(y)))$,

where $\alpha \neq 1$, $\alpha > 0$ and T_α are Frank's t-norms

$$T_\alpha(x,y) = \log_\alpha\,(1 + (\alpha^x - 1)(\alpha^y - 1)/(\alpha - 1))\,.$$

This theorem has a special value in our context since it shows how non-dual operations may appear and how Frank's family plays a relevant role.

8 Some final remarks

We have seen that there are, at least, good mathematical reasons to say that the most interesting basic triples (T,S,N) satisfying continuity and monotonic properties are **Frank's triples** and **normal triples**, depending the choice on the requirement of N-duality. But various examples have shown that if one forgets about continuity or monotonicity there are still many open problems: to characterize the various classes of triples. In some sense the realm of topological semigroups in closed real intervals has been essentially investigated. But without continuity, only pathological examples may be seen today. Since many membership functions have a finite number of values, such semigroups merit further research.

The study of the Hyers-Ulam stability of the equations giving the properties of the basic triples (T,S,N) may constitute also a rich field of analysis. In particular the characterization of operations T satisfying the inequality

$$|T(x, T(y,z)) - T(T(x,y),z)| \leq \epsilon,$$

for all x, y, z in [0,1] and for a given $\epsilon > 0$, may be an attractive problem in the agenda for the time to come.

Finally let us make some considerations on the "real role" of fuzzy connectives. In the classical boolean setting conjunctions, disjunctions and negations are used either to build new propositions or to perform set operations. If we look at the case of probability what is needed is the evaluation of the probability of unions or intersections or bounds for such probabilities, i.e., conjunctions and disjunctions are "measured". In the case of Statistics there is a special attention to random variables and their distributions and one studies operations between random variables as well as operations between distributions (being extremely important the study of joint distributions).

We believe that connectives in Fuzzy Logic present problems similar to those found in Statistics. For example, in many cases we can associate to vague predicates A, B some measures $m_A, m_B : X \to R^+$ and we can consider two fuzzy numbers $F_A, F_B : R^+ \to [0,1]$ and the corresponding membership functions $\mu_A(p) = F_A(m_A(p))$, $\mu_B(p) = F_B(m_B(p))$.