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HILBERT SPACE METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

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Preface

This book has developed from a course which we have given periodically over the last eight years. It is addressed to beginning graduate students of mathematics, engineering and the physical sciences. Thus, we have attempted to present it while presupposing a minimal background: the reader is assumed to have some prior acquaintance with the concepts of 'linear' and 'continuous' and also to believe L^2 is complete. An undergraduate mathematics training through Lebesgue integration is an ideal background but we dare not assume it without turning away many of our best students. The formal prerequisite consists of a good advanced calculus course and a motivation to study partial differential equations.

A problem is called *well-posed* if for each set of data there exists exactly one solution and this dependence of the solution on the data is continuous. To make this precise we must indicate the space from which the solution is obtained, the space from which the data may come, and the corresponding notion of continuity. Our goal in this book is to show that various types of problems are well-posed. These include boundary value problems for (stationary) elliptic partial differential equations and initial-boundary value problems for (time-dependent) equations of parabolic, hyperbolic, and pseudo-parabolic types. Also, we consider some nonlinear elliptic boundary value problems, variational or unilateral problems, and some methods of numerical approximation of solutions.

We shall briefly describe the contents of the various chapters. Chapter I presents all the elementary Hilbert space theory that is needed for the book. The first half of Chapter I is presented in a rather brief fashion and is intended both as a review for some readers and as a study guide for others. Non-standard items to note here are the spaces $C^m(\bar{G})$, V^*, and V'. The first consists of restrictions to the closure of G of functions on $\mathbb{R}^n$ and the last two consist of conjugate-linear functionals.

Chapter II is an introduction to distributions and Sobolev spaces. The latter are the Hilbert spaces in which we shall show various problems are well-posed. We use a primitive (and non-standard) notion of distribution which is adequate for our purposes. Our distributions are conjugate-linear and have the pedagogical advantage of being independent of any discussion of topological vector space theory.

Chapter III is an exposition of the theory of linear elliptic boundary value problems in variational form. (The meaning of 'variational form' is explained in Chapter VII.) We present an abstract Green's theorem which permits the separation of the abstract problem into a partial differential equation on the region and a condition on the boundary. This approach has the pedagogical advantage of making optional the discussion of regularity theorems. (We construct an operator ∂ which is an extension of the normal derivative on the boundary, whereas the normal derivative makes sense only for appropriately regular functions.)

Chapter IV is an exposition of the generation theory of linear semi-groups of contractions and its application to solve initial-boundary value problems for partial differential equations. Chapters V and VI provide the immediate extensions to cover evolution equations of second order and of implicit type. In addition to the classical heat and wave equations with standard boundary conditions, the applications in these chapters include a multitude of non-standard problems such as equations of pseudo-parabolic, Sobolev, viscoelasticity, degenerate or mixed type; boundary conditions of periodic or non-local type or with time-derivatives; and certain interface or even global constraints on solutions. We hope this variety of applications may arouse the interests even of experts.

Chapter VII begins with some reflections on Chapter III and develops into an elementary alternative treatment of certain elliptic boundary value problems by the classical Dirichlet principle. Then we briefly discuss certain unilateral boundary value problems, optimal control problems, and numerical approximation methods. This chapter can be read immediately after Chapter III and it serves as a natural place to begin work on nonlinear problems.

Each chapter is divided into sections and (usually) subsections. The notation 'Section III.5.2' refers to Chapter III, Section 5, Subsection 2. The notation 'Section 5.2' means Section 5, Subsection 2 of that chapter in which it occurs. Major results are labeled alphabetically within each section. Thus, 'Theorem III.5.C' means Theorem 5.C in Section 5 of Chapter III. In Chapter III we refer to it as Theorem 5.C. Displayed formulas and problems are referenced as (2.1) for Formula 1 of Section 2 in that same chapter; III(2.1) means (2.1) in Chapter III.

There are a variety of ways this book can be used as a text. In a year course for a well-prepared class, one may complete the entire book and supplement it with some related topics from nonlinear functional analysis.

In a semester course for a class with varied backgrounds, one may cover Chapters I, II, III, and VII. Similarly, with that same class one could cover in one semester the first four chapters. In any abbreviated treatment one could omit Sections I.6, II.4, II.5, III.6, the last three sections of Chapters IV, V, and VI, and Section VII.4. We have included over 40 examples in the exposition and there are about 200 exercises. The exercises are placed at the ends of the chapters and each is numbered so as to indicate the section for which it is appropriate.

Some suggestions for further study are arranged by chapter and precede the Bibliography. If the reader develops the interest to pursue some topic in one of these references, then this book will have served its purpose.

R. E. Showalter
Austin, Texas
January, 1977

Contents

I

Elements of Hilbert Space

1. Linear Algebra

We begin with some notation. A function F with domain $\text{dom}\,(F)=A$ and range $\text{Rg}\,(F)$ a subset of B is denoted by $F: A \to B$. That a point $x \in A$ is mapped by F to a point $F(x) \in B$ is indicated by $x \mapsto F(x)$. If S is a subset of A then the *image* of S by F is $F(S)=\{F(x): x \in S\}$. Thus $\text{Rg}(F)=F(A)$. The pre-image or inverse image of a set $T \subset B$ is $F^{-1}(T)=\{x \in A: F(x) \in T\}$. A function is called injective if it is one-to-one, surjective if it is onto, and bijective if it is both injective and surjective. Then it is called, respectively, an injection, surjection, or bijection.

$\mathbb{K}$ will denote the field of scalars for our vector spaces and is always one of $\mathbb{R}$ (real number system) or $\mathbb{C}$ (complex numbers). The choice in most situations will be clear from the context or immaterial, so we usually avoid mention of it.

The 'strong inclusion' $K \subset\subset G$ between subsets of Euclidean space $\mathbb{R}^n$ means K is compact, G is open, and $K \subset G$. If A and B are sets, their Cartesian product is given by $A \times B = \{[a, b]: a \in A, b \in B\}$. If A and B are subsets of $\mathbb{K}^n$ (or any other vector space) their set sum is $A+B=\{a+b: a \in A, b \in B\}$.

1.1

A *linear space* over the field $\mathbb{K}$ is a non-empty set V of vectors with a binary operation *addition* $+$: $V \times V \to V$ and a *scalar multiplication* $\cdot$: $\mathbb{K} \times V \to V$ such that $(V, +)$ is an Abelian group, i.e.,

$(x+y)+z=x+(y+z)$,	$x, y, z \in V$,
there is a zero $\theta \in V: x+\theta=x$,	$x \in V$,
if $x \in V$, there is $-x \in V: x+(-x)=\theta$,	and
$x+y=y+x$,	$x, y \in V$,

and we have

$$(\alpha+\beta)\cdot x=\alpha\cdot x+\beta\cdot x, \qquad \alpha\cdot(x+y)=\alpha\cdot x+\alpha\cdot y,$$
$$\alpha\cdot(\beta\cdot x)=(\alpha\beta)\cdot x, \qquad 1\cdot x=x, \qquad x, y\in V, \alpha, \beta\in\mathbb{K}.$$

We shall suppress the symbol for scalar multiplication since there is no need for it.

Examples (a) The set $\mathbb{K}^n$ of n-tuples of scalars is a linear space over $\mathbb{K}$. Addition and scalar multiplication are defined coordinatewise:

$$(x_1, x_2, \ldots, x_n)+(y_1, y_2, \ldots, y_n)=(x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$$
$$\alpha(x_1, x_2, \ldots, x_n)=(\alpha x_1, \alpha x_2, \ldots, \alpha x_n).$$

(b) The set $\mathbb{K}^X$ of functions $f: X\to\mathbb{K}$ is a linear space, where X is a non-empty set and we define $(f_1+f_2)(x)=f_1(x)+f_2(x)$, $(\alpha f)(x)=\alpha f(x)$, $x\in X$.

(c) Let $G\subset\mathbb{R}^n$ be open. The above pointwise definitions of linear operations give a linear space structure on the set $C(G, \mathbb{K})$ of continuous $f: G\to\mathbb{K}$. We normally shorten this to $C(G)$.

(d) For each n-tuple $\alpha=(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of non-negative integers, we denote by D^α the *partial derivative*

$$\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}$$

of order $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$. The sets $C^m(G)=\{f\in C(G): D^\alpha f\in C(G)$ for all $\alpha, |\alpha|\leq m\}$, $m\geq 0$, and $C^\infty(G)=\bigcap_{m\geq 1} C^m(G)$ are linear spaces with the operations defined above. We let D^θ be the identity where $\theta=(0, 0, \ldots, 0)$, so $C^0(G)=C(G)$.

(e) For $f\in C(G)$, the *support* of f is the closure in G of the set $\{x\in G: f(x)\neq 0\}$ and we denote it by $\underset{\sim}{f}$. $C_0(G)$ is the subset of those functions in $C(G)$ with compact support. Similarly, we define $C_0^m(G)=C^m(G)\cap C_0(G)$, $m\geq 1$ and $C_0^\infty(G)=C^\infty(G)\cap C_0(G)$.

(f) If $f: A\to B$ and $C\subset A$, we denote by $f|_C$ the *restriction* of f to C. We obtain useful linear spaces of functions on the closure $\bar{G}$ as follows:

$$C^m(\bar{G})=\{f|_{\bar{G}}: f\in C_0^m(\mathbb{R}^n)\}, \qquad C^\infty(\bar{G})=\{f|_{\bar{G}}: f\in C_0^\infty(\mathbb{R}^n)\}.$$

These spaces play a central role in our work below.

1.2

A subset M of the linear space V is a *subspace* of V if it is closed under the linear operations. That is, $x+y\in M$ whenever $x, y\in M$ and $\alpha x\in M$

for each $\alpha \in \mathbb{K}$ and $x \in M$. We denote that M is a subspace of V by $M \leq V$. It follows that M is then (and only then) a linear space with addition and scalar multiplication inherited from V.

Examples We have three chains of subspaces given by

$$\begin{aligned} C^j(G) &\leq C^k(G) \leq \mathbb{K}^G, \\ C^j(\bar{G}) &\leq C^k(\bar{G}), \qquad \text{and} \\ \{\theta\} \leq C_0^j(G) &\leq C_0^k(G), \qquad 0 \leq k \leq j \leq \infty. \end{aligned}$$

Moreover, for each k as above, we can identify $\varphi \in C_0^k(G)$ with that $\Phi \in C^k(\bar{G})$ obtained by defining Φ to be equal to φ on G and zero on ∂G, the boundary of G. Likewise we can identify each $\Phi \in C^k(\bar{G})$ with $\Phi|_G \in C^k(G)$. These identifications are 'compatible' and we have $C_0^k(G) \leq C^k(\bar{G}) \leq C^k(G)$.

1.3

We let M be a subspace of V and construct a corresponding *quotient space*. For each $x \in V$, define a *coset* $\hat{x} = \{y \in V : y - x \in M\} = \{x + m : m \in M\}$. The set $V/M = \{\hat{x} : x \in V\}$ is the *quotient set*. Any $y \in \hat{x}$ is a *representative* of the coset $\hat{x}$ and we clearly have $y \in \hat{x}$ if and only if $x \in \hat{y}$ if and only if $\hat{x} = \hat{y}$. We shall define addition of cosets by adding a corresponding pair of representatives and similarly define scalar multiplication. It is necessary first to verify that this definition is unambiguous.

Lemma *If $x_1, x_2 \in \hat{x}$, $y_1, y_2 \in \hat{y}$, and $\alpha \in \mathbb{K}$, then $\widehat{(x_1 + y_1)} = \widehat{(x_2 + y_2)}$ and $\widehat{(\alpha x_1)} = \widehat{(\alpha x_2)}$.*

The proof follows easily, since M is closed under addition and scalar multiplication, and we can define $\hat{x} + \hat{y} = \widehat{(x + y)}$ and $\alpha\hat{x} = \widehat{(\alpha x)}$. These operations make V/M a linear space.

Examples (a) Let $V = \mathbb{R}^2$ and $M = \{(0, x_2) : x_2 \in \mathbb{R}\}$. Then V/M is the set of parallel translates of the x_2-axis, M, and addition of two cosets is easily obtained by adding their (unique) representatives on the x_1-axis.

(b) Take $V = C(G)$. Let $x_0 \in G$ and $M = \{\varphi \in C(G) : \varphi(x_0) = 0\}$. Write each $\varphi \in V$ in the form $\varphi(x) = (\varphi(x) - \varphi(x_0)) + \varphi(x_0)$. This representation can be used to show that V/M is essentially equivalent (isomorphic) to $\mathbb{K}$.

(c) Let $V = C(\bar{G})$ and $M = C_0(G)$. We can describe V/M as a space of 'boundary values'. To do this, begin by noting that for each $K \subset\subset G$

there is a $\psi \in C_0(G)$ with $\psi = 1$ on K (cf. Section II.1.1). Then write a given $\varphi \in C(\bar{G})$ in the form

$$\varphi = (\varphi\psi) + \varphi(1-\psi),$$

where the first term belongs to M and the second equals φ in a neighborhood of ∂G.

1.4

Let V and W be linear spaces over $\mathbb{K}$. A function $T: V \to W$ is *linear* if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y), \qquad \alpha, \beta \in \mathbb{K}, \qquad x, y \in V.$$

That is, linear functions are those which preserve the linear operations. An *isomorphism* is a linear bijection. The set $\{x \in V : Tx = \theta\}$ is called the *kernel* of the (not necessarily linear) function $T: V \to W$ and we denote it by $K(T)$.

Lemma *If $T: V \to W$ is linear, then $K(T)$ is a subspace of V, $\mathrm{Rg}\,(T)$ is a subspace of W, and $K(T) = \{\theta\}$ if and only if T is an injection.*

Examples (a) Let M be a subspace of V. The identity $i_M : M \to V$ is a linear injection $x \mapsto x$ and its range is M.

(b) The *quotient* map $q_M : V \to V/M$, $x \mapsto \hat{x}$, is a linear surjection with kernel $K(q_M) = M$.

(c) Let G be the open interval (a, b) in $\mathbb{R}$ and consider $D \equiv d/dx : V \to C(\bar{G})$, where V is a subspace of $C^1(\bar{G})$. If $V = C^1(\bar{G})$, then D is a linear surjection with $K(D)$ consisting of constant functions on $\bar{G}$. If $V = \{\varphi \in C^1(\bar{G}) : \varphi(a) = 0\}$, then D is an isomorphism. Finally, if $V = \{\varphi \in C^1(\bar{G}) : \varphi(a) = \varphi(b) = 0\}$, then $\mathrm{Rg}\,(D) = \{\varphi \in C(\bar{G}) : \int_a^b \varphi = 0\}$.

Our next result shows how each linear function can be factored into the product of a linear injection and an appropriate quotient map.

Theorem 1.A *Let $T: V \to W$ be linear and M be a subspace of $K(T)$. Then there is exactly one function $\hat{T}: V/M \to W$ for which $\hat{T} \circ q_M = T$, and $\hat{T}$ is linear with $\mathrm{Rg}\,(\hat{T}) = \mathrm{Rg}\,(T)$. Finally, $\hat{T}$ is injective if and only if $M = K(T)$.*

Proof If $x_1, x_2 \in \hat{x}$, then $x_1 - x_2 \in M \subset K(T)$, so $T(x_1) = T(x_2)$. Thus we can define a function as desired by the formula $\hat{T}(\hat{x}) = T(x)$. The uniqueness and linearity of $\hat{T}$ follow since q_M is surjective and linear. The equality of the ranges follows, since q_M is surjective, and the last statement follows from the observation that $K(T) \subset M$ if and only if $x \in V$ and $\hat{T}(\hat{x}) = 0$ imply $\hat{x} = \hat{0}$.

An immediate corollary is that each linear function $T: V \to W$ can be factored into a product of a surjection, an isomorphism, and an injection:

$$T = i_{\mathrm{Rg}(T)} \circ \hat{T} \circ q_{K(T)}.$$

A function $T: V \to W$ is called *conjugate linear* if

$$T(\alpha x + \beta y) = \bar{\alpha} T(x) + \bar{\beta} T(y), \qquad \alpha, \beta \in \mathbb{K}, \qquad x, y \in V.$$

Results similar to those above hold for such functions.

1.5

Let V and W be linear spaces over $\mathbb{K}$ and consider the set $L(V, W)$ of linear functions from V to W. The set W^V of all functions from V to W is a linear space under the pointwise definitions of addition and scalar multiplication (cf. Example 1.1(b)), and $L(V, W)$ is a subspace.

We define V^* to be the linear space of all conjugate linear functionals from $V \to \mathbb{K}$. V^* is called the *algebraic dual* of V. Note that there is a bijection $f \mapsto \bar{f}$ of $\mathscr{L}(V, \mathbb{K})$ onto V^*, where $\bar{f}$ is the functional defined by $\bar{f}(x) = \overline{f(x)}$ for $x \in V$ and is called the *conjugate* of the functional $f: V \to \mathbb{K}$. Such spaces provide a useful means of constructing large linear spaces containing a given class of functions. We illustrate this technique in a simple situation.

Example Let G be open in $\mathbb{R}^n$ and $x_0 \in G$. We shall imbed the space $C(G)$ in the algebraic dual of $C_0(G)$. For each $f \in C(G)$, define $T_f \in C_0(G)^*$ by

$$T_f(\varphi) = \int_G f\bar{\varphi}, \qquad \varphi \in C_0(G).$$

Since $f\bar{\varphi} \in C_0(G)$, the Riemann integral is adequate here. An easy exercise shows that the function $f \mapsto T_f: C(G) \to C_0(G)^*$ is a linear injection, so we may thus identify $C(G)$ with a subspace of $C_0(G)^*$. This linear injection is not surjective; we can exhibit functionals on $C_0(G)$ which are not identified with functions in $C(G)$. In particular, the *Dirac functional* δ_{x_0} defined by

$$\delta_{x_0}(\varphi) = \overline{\varphi(x_0)}, \qquad \varphi \in C_0(G),$$

cannot be obtained as T_f for any $f \in C(G)$. That is, $T_f = \delta_{x_0}$ implies that $f(x) = 0$ for all $x \in G$, $x \neq x_0$, and thus $f = 0$, a contradiction.

2. Convergence and Continuity

The absolute value function on $\mathbb{R}$ and modulus function on $\mathbb{C}$ are denoted by $|\cdot|$, and each gives a notion of length or distance in the corresponding space and permits the discussion of convergence of sequences in that space or continuity of functions on that space. We shall extend these concepts to a general linear space.

2.1

A *seminorm* on the linear space V is a function $p: V \to \mathbb{R}$ for which $p(\alpha x) = |\alpha| p(x)$ and $p(x+y) \leqslant p(x) + p(y)$ for all $\alpha \in \mathbb{K}$ and $x, y \in V$. The pair V, p is called a *seminormed space.*

Lemma *If V, p is a seminormed space, then*

(a) $|p(x) - p(y)| \leqslant p(x-y)$, $x, y \in V$,
(b) $p(x) \geqslant 0$, $x \in V$, *and*
(c) *the kernel $K(p)$ is a subspace of V.*
(d) *If $T \in L(W, V)$, then $p \circ T: W \to \mathbb{R}$ is a seminorm on W.*
(e) *If p_j is a seminorm on V and $\alpha_j \geqslant 0$, $1 \leqslant j \leqslant n$, then $\sum_{j=1}^{n} \alpha_j p_j$ is a seminorm on V.*

Proof We have $p(x) = p(x-y+y) \leqslant p(x-y) + p(y)$ so $p(x) - p(y) \leqslant p(x-y)$. Similarly, $p(y) - p(x) \leqslant p(y-x) = p(x-y)$, so the result follows. Setting $y = \theta$ in (a) and noting $p(\theta) = 0$, we obtain (b). The result (c) follows directly from the definitions, and (d) and (e) are straightforward exercises.

If p is a seminorm with the property that $p(x) > 0$ for each $x \neq \theta$, we call it a *norm.*

Examples (a) For $1 \leqslant k \leqslant n$ we define seminorms on $\mathbb{K}^n$ by $p_k(x) = \sum_{j=1}^{k} |x_j|$, $q_k(x) = (\sum_{j=1}^{k} |x_j|^2)^{1/2}$, and $r_k(x) = \max\{|x_j| : 1 \leqslant j \leqslant k\}$. Each of p_n, q_n and r_n is a norm.

(b) If $J \subset X$ and $f \in \mathbb{K}^X$, we define $P_J(f) = \sup\{|f(x)| : x \in J\}$. Then for each finite $J \subset X$, P_J is a seminorm on $\mathbb{K}^X$.

(c) For each $K \subset\subset G$, P_K is a seminorm on $C(G)$. Also, $P_{\bar{G}} = P_G$ is a norm on $C(\bar{G})$.

(d) For each j, $0 \leqslant j \leqslant k$, and $K \subset\subset G$ we can define a seminorm on $C^k(G)$ by $p_{j,K}(f) = \sup\{|D^\alpha f(x)| : x \in K, |\alpha| \leqslant j\}$. Each such $p_{j,G}$ is a norm on $C^k(\bar{G})$.

2.2

Seminorms permit a discussion of convergence. We say the sequence $\{x_n\}$ in V *converges* to $x \in V$ if $\lim_{n\to\infty} p(x_n - x) = 0$; that is, if $\{p(x_n - x)\}$ is a sequence in $\mathbb{R}$ converging to 0. Formally, this means that for every $\varepsilon > 0$ there is an integer $N \geq 0$ such that $p(x_n - x) < \varepsilon$ for all $n \geq N$. We denote this by $x_n \to x$ in V, p and suppress the mention of p when it is clear what is meant.

Let $S \subset V$. The *closure* of S in V, p is the set $\bar{S} = \{x \in V : x_n \to x$ in V, p for some sequence $\{x_n\}$ in $S\}$ and S is called *closed* if $S = \bar{S}$. The closure $\bar{S}$ of S is the smallest closed set containing $S: S \subset \bar{S}$, $\bar{S} = \bar{\bar{S}}$, and if $S \subset K = \bar{K}$ then $\bar{S} \subset K$.

Lemma *Let V, p be a seminormed space and M be a subspace of V. Then $\bar{M}$ is a subspace of V.*

Proof Let $x, y \in \bar{M}$. Then there are sequences $x_n, y_n \in M$ such that $x_n \to x$ and $y_n \to y$ in V, p. But $p((x + y) - (x_n + y_n)) \leq p(x - x_n) + p(y - y_n) \to 0$ which shows that $(x_n + y_n) \to x + y$. Since $x_n + y_n \in M$, all n, this implies that $x + y \in \bar{M}$. Similarly, for $\alpha \in \mathbb{K}$ we have $p(\alpha x - \alpha x_n) = |\alpha| p(x - x_n) \to 0$, so $\alpha x \in \bar{M}$.

2.3

Let V, p and W, q be seminormed spaces and $T: V \to W$ (not necessarily linear). Then T is called *continuous* at $x \in V$ if for every $\varepsilon > 0$ there is a $\delta > 0$ for which $y \in V$ and $p(x - y) < \delta$ implies $q(T(x) - T(y)) < \varepsilon$. T is *continuous* if it is continuous at every $x \in V$.

Theorem 2.A *T is continuous at x if and only if $x_n \to x$ in V, p implies $Tx_n \to Tx$ in W, q.*

Proof Let T be continuous at x and $\varepsilon > 0$. Choose $\delta > 0$ as in the definition above and then N such that $n \geq N$ implies $p(x_n - x) < \delta$, where $x_n \to x$ in V, p is given. Then $n \geq N$ implies $q(Tx_n - Tx) < \varepsilon$, so $Tx_n \to Tx$ in W, q.

Conversely, if T is not continuous at x, then there is an $\varepsilon > 0$ such that for every $n \geq 1$ there is an $x_n \in V$ with $p(x_n - x) < 1/n$ and $q(Tx_n - Tx) \geq \varepsilon$. That is, $x_n \to x$ in V, p but $\{Tx_n\}$ does not converge to Tx in W, q.

We record the facts that our algebraic operations and seminorm are always continuous.