FUZZY SETS AND SYSTEMS

Theory and Applications

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FOREWORD

When I first met Henri Prade and Didier Dubois, I was impressed at once by their unusual breadth of knowledge about all facets of the theory of fuzzy sets and their youthful enthusiasm for a theory that challenges the traditional reliance on two-valued logic and classical set theory as a basis for scientific inquiry.

Later on, when they told me about their plans for writing an up-to-date research monograph on fuzzy sets and systems, I was rather skeptical that it could be done although the earlier five-volume work of Professor Arnold Kaufmann had covered the basic ground both comprehensively and with great authority.

The publication of this volume shows that my skepticism was unwarranted. Dubois and Prade have produced a comprehensible research monograph that covers almost all of the important developments in the theory of fuzzy sets and in their applications that have taken place during the past several years—developments that include their own significant contributions to fuzzy arithmetic and the analysis of fuzzy relations.

In presenting the work of others, Dubois and Prade have contributed many useful insights and supplied a number of examples which aid materially in understanding of the subject matter. Inevitably, there are some instances where one could take issue with their choice of topics, their interpretations, and their conclusions. But what is remarkable is that they have been able to cover so much ground—within the compass of a single volume—in a field that is undergoing rapid growth and spans a wide variety of applications, ranging from industrial process control to medical diagnosis and group decision processes.

Like other theories that have broken away from tradition, the theory of fuzzy sets has been and will continue to be controversial for some time to come. The present volume may or may not convince the skeptics of the utility of fuzzy sets. But it will certainly be of great value to those who are interested in acquainting themselves with the basic aspects of the theory and in exploring its potentialities as a methodology for dealing with phenomena that are too complex or too ill-defined to be susceptible to analysis by conventional means.

LOTFI A. ZADEH

PREFACE

Since Lotfi A. Zadeh published his now classic paper almost fifteen years ago, fuzzy set theory has received more and more attention from researchers in a wide range of scientific areas, especially in the past few years. This theory is attractive because it is based on a very intuitive, although somewhat subtle, idea capable of generating many intellectually appealing results that provide new insights to old, often-debated questions. Opinion is still divided about the importance of fuzzy set theory. Some people have argued that many contributions were simply exercises in generalization. However, several significant and original developments have recently been proposed, which should convince those who are still reluctant. Anyway, fuzziness is not a matter of aesthetics; neither is it an ingredient to make up arid formal constructions; it is an unavoidable feature of most humanistic systems and it must be dealt with as such.

This book is intended to be a rather exhaustive research monograph on fuzzy set theory and its applications. The work is based on a large compilation of the literature* in English, French, and German. Approximately 550 publications or communications† are referred to; it is hoped that they are representative of about a thousand papers existing in the world. Whenever possible we have tried to cite published easy-to-find versions of works rather than rare research memoranda. Of course, some original contributions may have been missed; this is unavoidable in such a fast growing field of research.

It is not intended here to embed fuzzy set theory in a pure mathematics framework. Sophisticated formalisms, such as that of category theory, do not seem suitable in working with concepts at an early stage of their development. No high-level mathematical tool will be used in the exposition.

We do not propose that this work be used as a textbook, but only as a research compendium. As such, topics are developed unequally according

*Throughout this book NF stands for references to the nonfuzzy literature.
†Appearing between 1965 and 1978.

to our own state of knowledge and fields of interest. Hence some chapters are only modest surveys of existing works, while others may appear more original and detailed. More specifically, there are very few tutorial numerical examples and no exercises; however, some hints or ideas at their early stage of development can be found, which we hope will be of some use for further research.

This book is a structured synthesis in an attempt to unify existing works. Such an attempt is made necessary because several research directions have been investigated, often independently.

In spite of the relative lack of mathematical ambition within the work, some may find the material rather hard to read because it covers a wide range of topics within a comparatively small number of pages. Thus, this monograph is aimed at readers at the graduate level, involved in research dealing with human-centered systems.

This synthesis is organized in five parts respectively devoted to (1) a short informal discussion on the nature of fuzziness; different kinds of uncertainty are pointed out; (2) a structured exposition in five chapters of the mathematics of fuzzy sets; (3) a description of fuzzy models and formal structures: logic, systems, languages and algorithms, and theoretical operations research; (4) a survey of system-oriented applied topics dealing with fuzzy situation; (5) a brief review of results in existing fields of applications.

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Part I
INTRODUCTION

Fuzziness is not a priori an obvious concept and demands some explanation. "Fuzziness" is what Black (NF 1937) calls "vagueness"† when he distinguishes it from "generality" and from "ambiguity." Generalizing refers to the application of a symbol to a multiplicity of objects in the field of reference, ambiguity to the association of a finite number of alternative meanings having the same phonetic form. But, the fuzziness of a symbol lies in the lack of well-defined boundaries of the set of objects to which this symbol applies.

More specifically, let X be a field of reference, also called a universe of discourse or universe for short, covering a definite range of objects. Consider a subset $\tilde{A}$ where transition between membership and nonmembership is gradual rather than abrupt. This "fuzzy subset" obviously has no well-defined boundaries. Fuzzy classes of objects are often encountered in real life. For instance, $\tilde{A}$ may be the set of tall men in a community X. Usually, there are members of X who are definitely tall, others who are definitely not tall, but there exist also borderline cases. Traditionally, the grade of membership 1 is assigned to the objects that completely belong to $\tilde{A}$—here the men who are definitely tall; conversely the objects that do not belong to $\tilde{A}$ at all are assigned a membership value 0. Quite naturally, the grades of membership of the borderline cases lie between 0 and 1. The

† However, it must be noticed that Zadeh (1977a) [Reference from IV.2] has used the word "vagueness" to designate the kind of uncertainty which is *both* due to fuzziness and ambiguity.

more an element or object x belongs to $\tilde{A}$, the closer to 1 is its grade of membership $\mu_{\tilde{A}}(x)$. The use of a numerical scale such as the interval [0, 1] allows a convenient representation of the gradation in membership. Precise membership values do not exist by themselves, they are tendency indices that are subjectively assigned by an individual or a group. Moreover, they are context-dependent. The grades of membership reflect an "ordering" of the objects in the universe, induced by the predicate associated with $\tilde{A}$; this "ordering," when it exists, is more important than the membership values themselves. The membership assessment of objects can sometimes be made easier by the use of a similarity measure with respect to an ideal element. Note that a membership value $\mu_{\tilde{A}}(x)$ can be interpreted as the degree of compatibility of the predicate associated with $\tilde{A}$ and the object x. For concepts such as "tallness," related to a physical measurement scale, the assignment of membership values will often be less controversial than for more complex and subjective concepts such as "beauty."

The above approach, developed by Zadeh (1964), provides a tool for modeling human-centered systems (Zadeh, 1973). As a matter of fact, fuzziness seems to pervade most human perception and thinking processes. Parikh (1977) has pointed out that no nontrivial first-order-logic-like observational predicate (i.e., one pertaining to perception) can be defined on an observationally connected space;† the only possible observational predicates on such a space are not classical predicates but "vague" ones. Moreover, according to Zadeh (1973), one of the most important facets of human thinking is the ability to summarize information "into labels of fuzzy sets which bear an approximate relation to the primary data." Linguistic descriptions, which are usually summary descriptions of complex situations, are fuzzy in essence.

It must be noticed that fuzziness differs from imprecision. In tolerance analysis imprecision refers to lack of knowledge about the value of a parameter and is thus expressed as a crisp tolerance interval. This interval is the set of possible values of the parameters. Fuzziness occurs when the interval has no sharp boundaries, i.e., is a fuzzy set $\tilde{A}$. Then, $\mu_{\tilde{A}}(x)$ is interpreted as the degree of possibility (Zadeh, 1978) that x is the value of the parameter fuzzily restricted by $\tilde{A}$.

The word *fuzziness* has also been used by Sugeno (1977) in a radically different context. Consider an arbitrary object x of the universe X; to each nonfuzzy subset A of X is assigned a value $g_x(A) \in [0, 1]$ expressing the

† Let $\alpha > 0$. A metric space is α-connected if it cannot be split into two disjoint nonempty ordinary subsets A and B such that $\forall x \in A$, $\forall y \in B$, $d(x, y) > \alpha$, where d is a distance. A metric space is observationally connected if it is α-connected for some α smaller than the perception threshold.

"grade of fuzziness" of the statement "x belongs to A." In fact this grade of fuzziness must be understood as a grade of *certainty*: according to the mathematical definition of g, $g_x(A)$ can be interpreted as the probability, the degree of subjective belief, the possibility, that x belongs to A. Generally, g is assumed increasing in the sense of set inclusion, but not necessarily additive as in the probabilistic case. The situation modeled by Sugeno is more a matter of guessing whether $x \in A$ rather than a problem of vagueness in the sense of Zadeh. The existence of two different points of view on "fuzziness" has been pointed out by MacVicar-Whelan (1977) and Skala (Reference from III.1). The monotonicity assumption for g seems to be more consistent with human guessing than does the additivity assumption. Moreover, grades of certainty can be assigned to fuzzy subsets $\tilde{A}$ of X owing to the notion of a fuzzy integral (see II.5.A.b). For instance, seeing a piece of Indian pottery in a shop, we may try to guess whether it is genuine or counterfeit; obviously, genuineness is not a fuzzy concept. x is the Indian pottery; A is the crisp set of genuine Indian artifacts; and $g_x(A)$ expresses, for instance, a subjective belief that the pottery is indeed genuine. The situation is slightly more complicated when we try to guess whether the pottery is old: actually, the set $\tilde{A}$ of old Indian pottery is fuzzy because "old" is a vague predicate.

It will be shown in III.1 that the logic underlying fuzzy set theory is multivalent. Multivalent logic can be viewed as a calculus either on the level of credibility of propositions or on the truth values of propositions involving fuzzy predicates. In most multivalent logics there is no longer an excluded-middle law; this situation may be interpreted as either the absence of decisive belief in one of the sides of an alternative or the overlapping of antonymous fuzzy concepts (e.g., "short" and "tall").

Contrasting with multivalent logics, a fuzzy logic has been recently introduced by Bellman and Zadeh (Reference from III.1). "Fuzzy logic differs from conventional logical systems in that it aims at providing a model for approximate rather than precise reasoning." In fuzzy logic what matters is not necessarily the calculation of the absolute (pointwise) truth values of propositions; on the contrary, a fuzzy proposition induces a possibility distribution over a universe of discourse. Truth becomes a relative notion, and "true" is a fuzzy predicate in the same sense as, for instance, "tall."

As an example, consider the proposition "John is a tall man." It can be understood in several ways. First, if the universe is a set of men including John and the set of tall men is a known fuzzy set $\tilde{A}$, then the truth-value of the proposition "John is a tall man" is $\mu_{\tilde{A}}$(John). Another situation consists in guessing whether John, about whom only indirect information is available, is a tall man; the degree of certainty of the proposition is expressed by $g_{\mathrm{John}}(\tilde{A})$. In contrast, in fuzzy logic we take the proposition "John is a tall man" as assumed, and we are interested in determining the information it conveys. "Tall" is then in a universe of heights a known fuzzy set that fuzzily restricts John's height. In other words, "John is a tall man" translates into a possibility distribution $\pi = \mu_{\mathrm{tall}}$. Then $\mu_{\mathrm{tall}}(h)$ gives a value to the possibility that John's height is equal to h. The possibility that John's height lies in the interval $[a,b]$ is easily calculated as

$$g_{\mathrm{John}}([a,b]) = \sup_{a \leq h \leq b} \mu_{\mathrm{tall}}(h),$$

as explained in II.5.B. It can also be verified, using a fuzzy integral, that $g_{\mathrm{John}}(\mathrm{tall}) = 1$, when "tall" is normalized (see II.1.A). This is consistent with taking the proposition "John is a tall man" as assumed.

One of the appealing features of fuzzy logic is its ability to deal with approximate causal inferences. Given an inference scheme "if P, then Q" involving fuzzy propositions, it is possible from a proposition P' that matches only approximately P, to deduce a proposition Q' approximately similar to Q, through a logical interpolation called "generalized modus ponens." Such an inference is impossible in ordinary logical systems.

APPENDIX: SOME HISTORICAL AND BIBLIOGRAPHICAL REMARKS

Fuzzy set theory was initiated by Zadeh in the early 1960s (1964; see also Bellman *et al.*, 1964). However, the term *ensemble flou* (a posteriori the French counterpart of *fuzzy set*) was coined by Menger (1951) in 1951. Menger explicitly used a "max-product" transitive fuzzy relation (see II.3.B.c.β), but with a probabilistic interpretation. On a semantic level Zadeh's theory is more closely related to Black's work on vagueness (Black, NF 1937), where "consistency profiles" (the ancestors of fuzzy membership functions) "characterize vague symbols."

Since 1965, fuzzy set theory has been considerably developed by Zadeh himself and some 300 researchers. This theory has begun to be applied in a wide range of scientific areas.

There have already been two monographs on fuzzy set theory published: a tutorial treatise in several volumes by Kaufmann (1973, 1975a, b, 1977; and others in preparation) and a mathematically oriented concise book by Negoita and Ralescu (1975). There are also two collections of papers edited by Zadeh *et al.* (1975) and Gupta *et al.* (1977).

Apart from Zadeh's excellent papers, other introductory articles are those of Gusev and Smirnova (1973), Ponsard (Reference from II.1), Ragade and Gupta (Reference from II.1), and Kandel and Byatt (1978). Rationales and discussions can also be found in Chang (1972), Ponsard

(1975), Sinaceur (1978), Gale (1975), Watanabe (1969, 1975), and Aizerman (1977).

Several bibliographies on fuzzy sets are available in the literature, namely, those of De Kerf (1975), Kandel and Davis (1976), Gaines and Kohout (1977), and Kaufmann (1979).

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