KINEMATICS AND DYNAMICS OF PLANE MECHANISMS

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PREFACE

"Numerous fields of engineering require new and, if possible, automatic machinery as well as complicated instruments and other apparatus, the creation of which requires not only great ingenuity, but also a thorough knowledge of mechanical principles. In this respect, one of the foremost and fundamental branches of engineering science is the science of kinematics and mechanisms, without which the creation of such devices can hardly be attempted."[1]

These words were written in 1941, when kinematics of mechanisms was in the doldrums, at least in the United States. Their truth was widely recognized at the time, and today, after twenty years of vigorous research activity, which continuously gathers momentum, kinematics of mechanisms has become a major discipline in its own right.

Unfortunately, though perhaps understandably, the related textbook literature has not kept pace with the developments in the field. This book was written in an attempt to help bridge the existing gap. Although it is primarily intended for undergraduate courses, a substantial part of it is of graduate standard. The prerequisites for both the graduate and undergraduate sections are sophomore Mathematics and Mechanics.

Much of the subject matter, gathered from various engineering journals, is of recent origin, and some is based on hitherto unpublished work. Emphasis throughout is placed on the graphical, rather than on the analytical, approach, because it is the more straightforward of the two, and yields results of an acceptable accuracy, except in those rare cases which require rigorously exact answers.

Although, in collating the material into an integrated whole, I found it necessary to modify nomenclature and notation, readers wishing to consult the original publications should experience little difficulty on this score.

In general, the selection of the subject matter and its presentation reflect my educational background, professional experience, and teaching philosophy. I should feel amply rewarded if the book succeeded in

[1] A. E. R. de Jonge, What is wrong with "Kinematics" and "Mechanisms"?, *Mech. Eng.*, vol. 64, no. 4, pp. 273–278, 1942.

awakening the interest of the students and stimulating them to further study of this fascinating subject.

Because of the great diversity of undergraduate mechanical-engineering courses, with their varying degrees of orientation either toward the applied or the theoretical aspects of engineering, it is not possible to make definite suggestions regarding a suitable allocation of place and time to Kinematics of Mechanisms within the curriculum.

The following scheme has been adopted by the School of Mechanical Engineering at the University of New South Wales. The topics reviewed in Chapters 1, 2, 3, and 7 of this book are treated in the first- and second-year courses in Mechanics (Statics and Dynamics); Chapters 4, 5, and 8 form part of the first course in Theory of Machines, given in the third year; Chapters 6 and 9 and selected sections of Chapter 10 form part of the second course in Theory of Machines, included in the curriculum of the fourth (final) year of the undergraduate honors course; and the balance of Chapter 10 and Chapters 11, 12, and 13 form the graduate subject Advanced Kinematics of Mechanisms. The time allocations (lectures plus practical work) for the topics mentioned are 40 hours in the third and fourth years and 90 hours for the graduate course.

I wish to express my sincere thanks to the Editorial Staffs of the American Society of Mechanical Engineers, the Institution of Mechanical Engineers, *Machine Design*, and *Product Engineering* for permission to draw upon material published in their respective journals; to Professor A. S. Hall, Jr., of Purdue University and to Professor F. Freudenstein of Columbia University for the generous loan of their lecture notes; and to my friends Professors F. W. Ocvirk and R. M. Phelan of Cornell University for their support and encouragement in the early stages of the project.

J. Hirschhorn

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INTRODUCTION

The design of a new machine or device for the performance of an operation, or sequence of operations, associated with some particular industrial process, usually involves the following steps:

1. An assessment of the problem
2. A conception of a suitable mechanism in its skeletal form
3. A kinematic analysis, or examination of the mechanism's motion characteristics from a purely geometrical point of view, which may reveal the need for a modification of the layout
4. A static analysis, or determination of the nature and magnitude of the forces associated with the primary function of the device
5. A choice of suitable materials of construction, based on technological and economic considerations, and a tentative proportioning of the members
6. A dynamic analysis, or determination of the inertia forces and their effects on safety and operational requirements, which may disclose the need for redesign

The chief purpose of this book is to provide the student with the proper tools for carrying out steps 3, 4, and 6 and to give him a basis for a rational approach to some problems of synthesis. It is also hoped that the book will prove a useful source of reference to the practicing engineer.

Before proceeding to the detailed investigation of the kinematic and dynamic behavior of mechanisms, it will be necessary to select a suitable and consistent system of units and it will be advisable to review some fundamental notions, usually discussed in basic courses in mathematics, physics, or general mechanics.

Because the concept of *force* is of more immediate interest to the engineer than that of *mass*, force is chosen as one of the three fundamental quantities in the engineering or gravitational system of units, the other two being *displacement* and *time*. The fundamental units of measure adopted in this book are, respectively, the *pound* (lb), the *inch* (in.), and the *second* (sec). The reasons for selecting the inch, rather than the foot, as unit of displacement are threefold:

1. Relative displacements of machine parts are generally of the order of a few inches, and sometimes amount to only fractions of an inch.

2. Dimensions of machine elements are usually given in inches.

3. Quantities used in the analysis of stress and strain are based on the inch, e.g., modulus of elasticity (lb/in.^{2}) and moment of inertia of cross section (in.^{4}). The accompanying table lists the most important quantities and their units of measure.

ENGINEERING SYSTEM OF UNITS

Quantity	Symbol	Unit
Displacement	s, x	in.
Time	τ	sec
Force	F, R	lb
Velocity, speed	v, $\dot{s}$, $\dot{x}$	in./sec
Acceleration	a, $\ddot{s}$, $\ddot{x}$	in./sec^2
Angular displacement	θ, ϕ	rad
Angular velocity	ω	rad/sec
Angular acceleration	α	rad/sec^2
Mass	M	lb-sec^2/in.
Linear momentum	$\mathcal{M}$	lb-sec
Torque	T	in.-lb†
Moment of inertia	I	lb-in.-sec^2
Angular momentum	$\mathcal{J}$	in.-lb-sec
Work	$\mathcal{W}$	lb-in.†
Energy	$\mathcal{E}$	lb-in.
Power	$\mathcal{P}$	lb-in./sec

† The reasons for choosing the in.-lb as unit of torque and the lb-in. as unit of work are given in Secs. 1-4 and 1-5.

CHAPTER 1

FUNDAMENTALS OF VECTOR ANALYSIS

1-1. Scalars and Vectors

Physical quantities are divided into *scalars* and *vectors*. Scalars, examples of which are mass, time, and work, are completely defined by magnitude and units of measure. Vectors, such as force, velocity, and acceleration, require, in addition, the specification of direction.

Provided that the physical nature of vector quantities is kept in mind, vector analysis, or mathematical manipulation of vectors, becomes a powerful tool in the investigation of many physical phenomena and helps greatly in their proper understanding.

1-2. Vector Notation; Unit Vector; Representation of Vectors

Vector Notation. Vector quantities will be denoted by bold-faced symbols; their magnitudes, and scalar quantities in general, will be designated by italics:

$$\mathbf{F},\ F;\ \mathbf{r},\ r;\ \mathbf{v},\ v;\ \text{etc.}$$

Unit Vector. A very useful concept in vector analysis is the *unit vector*, a directed element of length *one*, having no physical units. It will be denoted by the symbol **i** with an appropriate lower-case suffix:

$\mathbf{i}_s$ denotes a unit vector in the direction s

Representation of Vectors. A vector quantity is depicted conveniently by a directed line segment, or arrow, of a length representing, to some suitable scale, the actual magnitude. In the case of translative or lineal vectors, such as force or velocity, the line vector is drawn parallel to the line of action of the quantity considered and pointing in its direction. In depicting rotational vectors, such as torque or angular velocity, the line vector is usually[1] drawn parallel to the axis around which the action takes place, pointing in the direction in which a right-hand screw would

[1] When dealing with the balancing of rotating and reciprocating masses, a different convention is adopted.

advance if turned in the sense of the particular quantity. In the special case of a two-dimensional system, the simple device of a curved arrow is

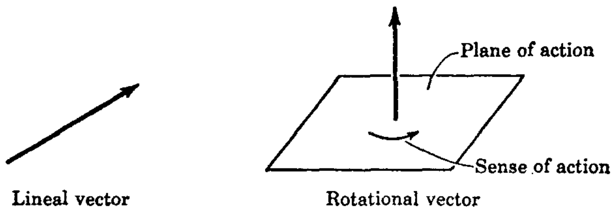

FIG. 1-1

frequently used to indicate the sense as clockwise (cw) or counterclockwise (ccw), and the magnitude is stated separately.

1-3. Composition, Subtraction, and Resolution of Vectors

Resultant. By definition, the *resultant* of a vector system is a vector obtained by the process of *composition*, or *geometrical* addition.

Parallelogram Method of Composition. The resultant **s** of two vectors **a** and **b** is constructed by setting off the vectors from a common origin, or pole, and then completing the parallelogram, as shown in Fig. 1-2. The diagonal which originates at the pole represents the resultant in magnitude and direction.

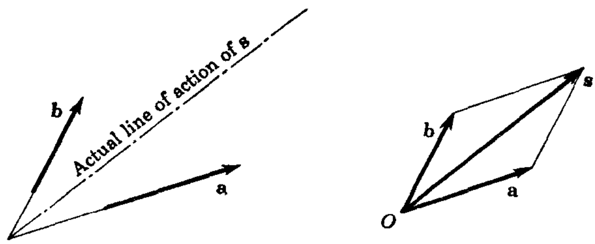

FIG. 1-2

The construction may be extended progressively to any number of vectors. In Fig. 1-3, the resultant **s** of **a**, **b**, and **c** is found by first constructing the resultant **s′** of **a** and **b** and then combining it with **c**.

Polygon Method of Composition. Examination of Fig. 1-3 reveals that the same result would be obtained by arranging the individual vectors "in order," i.e., tail to tip, and then joining the initial point, or tail, of the first vector with terminal point, or tip, of the last vector. Furthermore, it is evident that the sequence in which the vectors are taken has no effect on the result; i.e., the commutative law is valid in the geometri-

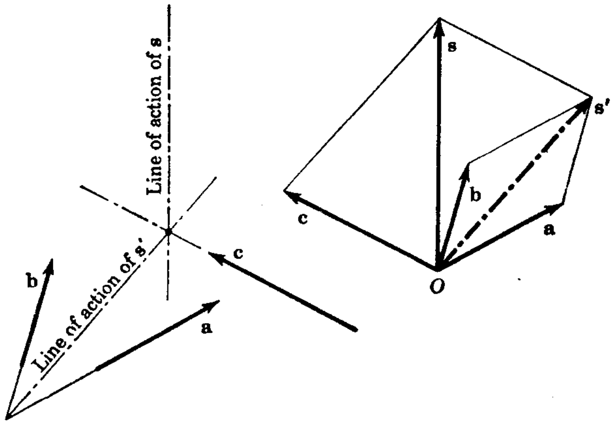

FIG. 1-3

cal addition of vectors, just as it is in the algebraic addition of scalars:

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{b} + \mathbf{a} + \mathbf{c}, \text{ etc.}$$

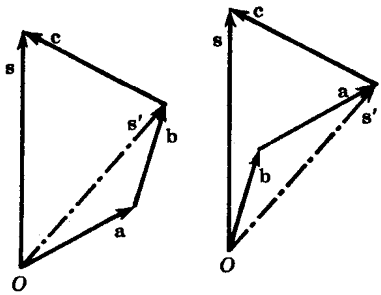

FIG. 1-4

Free, Sliding, and Fixed Vectors. It is important to realize that, although in the mathematical sense two vectors are considered equal if they have the same magnitude and direction, they may produce, or be the result of, different physical effects. (For instance, two equal parallel forces applied in turn to a given rigid body would produce the same acceleration of its center of gravity but different angular accelerations.) For this reason, three types of vector are distinguished in mechanics, viz., the free, the sliding, and the fixed. A *free vector* is tied neither to a specific point of application nor to a particular line of action. The velocity and acceleration of a rigid body in translation are examples of free vectors, because in this type of motion all particles have the same motion characteristics. A *sliding vector* is tied only to a specific line of action. The dynamic effect of a force acting on a body is not affected by a displacement of the force along its line of action. Hence, in dynamics, forces are regarded as sliding vectors. Another example of a sliding vector is the following. In a rigid body the distances between the particles do not change. Consequently, collinear particles of such a body have the same velocity component in the direction of the line containing them. Thus this velocity component is a sliding vector. A *fixed vector* is tied to a particular point of application. The velocity of a specific point or particle is an example of a fixed vector. A second example is illustrated in Fig. 1-5.

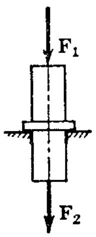

FIG. 1-5

Although $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal and act along the same line, their static effects differ: $\mathbf{F}_1$ causes compression of the upper part of the pin, while $\mathbf{F}_2$ produces tensile stresses in the portion below the collar. Hence, if the distribution of direct stresses due to a given force is to be investigated, the force must be regarded as a fixed vector.

Position of the Resultant of Concurrent Vectors. The resultant of a system of concurrent vectors passes through the common point of intersection.

Position of the Resultant of Coplanar Nonconcurrent Vectors. The resultant of a system of coplanar nonconcurrent vectors has physical significance only if the vectors concerned are forces or momenta. As indicated in Fig. 1-3, its line of action may be found by a gradual addition of the sliding component vectors. Alternatively, the line of action may be determined from the condition that the torque of the resultant about any point in the plane is equal to the algebraic sum of the torques of the component vectors with respect to the same reference point. Experience shows that the effect of the resultant is equal to the combined effect of the original system. For instance, a body in motion under the action of a two-dimensional force system will acquire a definite acceleration of its center of gravity and a definite angular acceleration. Identical dynamic effects would also be produced by a single force, equal to the resultant, applied along the proper line of action.

Subtraction of Vectors. The subtraction of a vector is equivalent to the addition of its negative, i.e., of a vector having the same magnitude but opposite direction:

$$\mathbf{d} = \mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

Resolution of Vectors. The individual vectors which together form the resultant are called its components. *Resolution* of a given vector is the process of finding its components in specified directions.

A vector can be resolved uniquely into only two related components. If more than two directions are prescribed, the number of possible combinations of components becomes infinite.

In determining the components, the procedure of the parallelogram method is reversed, as shown in Fig. 1-6.

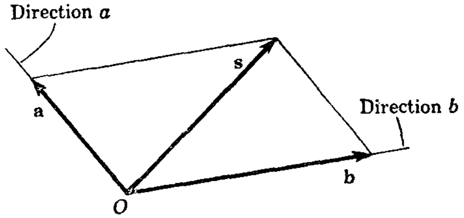

FIG. 1-6

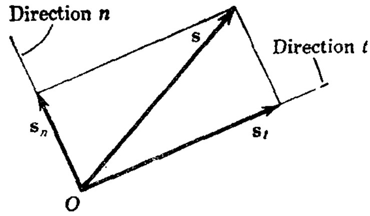

FIG. 1-7

Of great practical importance are mutually perpendicular components, or *projections*, of a vector. Projections will be denoted by the same letter as the vector, with appropriate lower-case suffixes, as shown in Fig. 1-7.

Occasionally, as in the kinematic analysis of complex mechanisms, a vector may be specified by two projections in *independent* directions. The corresponding construction of the vector is shown in Fig. 1-8. The difference between it and the ordinary composition should be noted carefully.

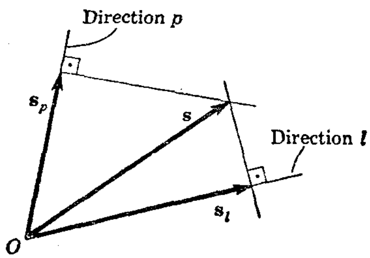

FIG. 1-8

Analytical Composition of Vectors. The vectors are referred to a cartesian system of coordinates, and their components in the x and y directions are calculated as shown in general terms for the vector **q**:

$$q_x = q \cos \theta_q \qquad \text{and} \qquad q_y = q \sin \theta_q \tag{1-1}$$

The components s_x and s_y of the resultant are computed by adding algebraically the corresponding component projections:

$$s_x = \Sigma q_x \qquad \text{and} \qquad s_y = \Sigma q_y \tag{1-2}$$

The magnitude of the resultant is given by

$$s = (s_x^2 + s_y^2)^{\frac{1}{2}} \tag{1-3}$$

and its directional angle is calculated from

$$\tan \theta_s = \frac{s_y}{s_x} \tag{1-4}$$

In order to obtain the correct algebraic signs of the components, the directional angles are measured in the counterclockwise sense from the positive x axis, as indicated in Fig. 1-9.

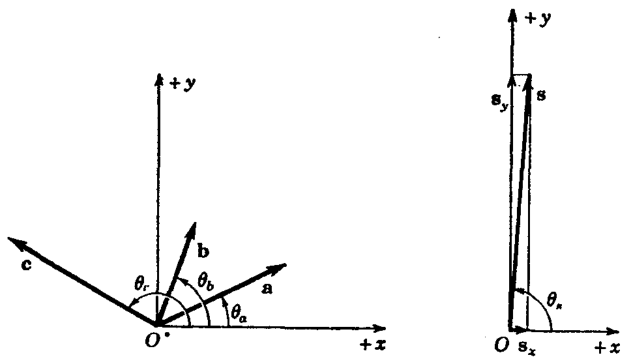

FIG. 1-9

1-4. Multiplication of a Vector by a Pure Number or a Scalar

From a formalistic point of view the product $M\mathbf{a}$ is a vector which has the same direction as $\mathbf{a}$ and a magnitude M times as great, irrespective of whether M is a pure number or a scalar. From the physical point of view, however, there is a considerable difference in the nature of the two resulting vectors, because a pure number does not change the physical character of the original vector, while a scalar quantity does: if, in the product $\mathbf{A} = M\mathbf{a}$, M is a pure number and $\mathbf{a}$ an acceleration, then $\mathbf{A}$ is also an acceleration; if, however, M represents *mass*, then $\mathbf{A}$ becomes a force.

It follows, therefore, that a vector quantity may be expressed as the product of its magnitude (a scalar) and the appropriate unit vector; e.g.,

$$\mathbf{q} = m\mathbf{i}_r$$

signifying that the vector $\mathbf{q}$ has a magnitude m and acts in the direction r. (It is not necessary to denote the magnitude and direction by the same letter as the vector itself.)

1-5. The Vectorial, or "Cross," Product of Two Vectors

The vectorial product of $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a} \times \mathbf{b}$, is defined as a rotational vector of magnitude ($ab \sin \theta_{ab}$), normal to the plane of $\mathbf{a}$ and