STATISTICAL FIELD THEORY

Volume 2

Strong coupling, Monte Carlo methods, conformal field theory, and random systems

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CAMBRIDGE UNIVERSITY PRESS

Cambridge

New York   *Port Chester*   *Melbourne*   *Sydney*

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
40 West 20th Street, New York, NY 10011, USA
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1989

Printed in the United States of America

Library of Congress Cataloging-in-Publication Data

Itzykson, Claude.
Statistical field theory / Claude Itzykson, Jean-Michel Drouffe.
p. cm. – (Cambridge monographs on mathematical physics)
Includes bibliographies and indexes.
Contents: v. 1 . From Brownian motion to renormalization and lattice gauge theory – v. 2. Strong coupling, Monte Carlo methods, conformal field theory, and random systems.
ISBN 0-521-34058-6 (v. 1). ISBN 0-521-37012-4 (v. 2)
1. Statistical physics. 2. Field theory (Physics) I. Drouffe, Jean-Michel. II. Title. III. Series.
QC174.8.I89 1989
530.1'3—dc19 88-30162
CIP

British Library Cataloging in Publication applied for.

Preface

Some ten years ago, when completing with J.-B. Zuber a previous text on *Quantum Field Theory*, the senior author was painfully aware that little mention was made that methods in statistical physics and Euclidean field theory were coming closer and closer, with common tools based on the use of path integrals and the renormalization group giving insights on global structures. It was partly to fill this gap that the present book was undertaken. Alas, over the five years that it took to come to life, both subjects have undergone a new evolution. Disordered media, growth patterns, complex dynamical systems or spin glasses are among the new important topics in statistical mechanics, while superstring theory has turned to the study of extended systems, Kaluza–Klein theories in higher dimensions, anticommuting coordinates ... in an attempt to formulate a unified model including all known interactions. New and sophisticated techniques have invaded statistical physics, ranging from algebraic methods in integrable systems to fractal sets or random surfaces. Powerful computers or special devices provide "experimental" means for a new brand of theoretical physicists. In quantum field theory, applications of differential topology, geometry, Riemannian manifolds, operator theory ... require a deeper background in mathematics and a knowledge of some of its most recent developments. As a result, when surveying what has been included in the present volume in an attempt to uncover the basic unity of these subjects, the authors have the same unsatisfactory feeling of not being able to bring the reader really up to date. It is presumably the fate of such endeavours to always come short of accomplishing their purpose.

With these shortcomings fully admitted, we have tried to present to the reader an overview of the main themes which justify the title "Statistical field theory." This interpretation of Euclidean field theory offers a new language, effective computing means, as well as a natural and consistent short-distance cutoff. In other words, it allows one to give a global meaning to path integrals, to discover possible anomalies arising from integration measures, or to understand in simple terms systems with redundant variables such as gauge models. The theory of continuous phase transitions provides a bridge between probabilistic mechanics and continuous field theory, using the renormalization group to filter out relevant operators and interactions. Many authors contributed to these views, culminating in the work of K. Wilson and his collaborators and followers, which promoted the renormalization group as a universal tool to analyse the large-distance behaviour. It still retains its value, while new developments take place, particularly with conformal, or local scale invariance coming to prominence in the study of two-dimensional systems.

The content of this book is naturally divided into two parts. The first six chapters describe in succession Brownian motion, its anticommutative counterpart in the guise of Onsager's solution to the two-dimensional Ising model, the mean-field or Landau approximation, scaling ideas exemplified by the Kosterlitz–Thouless theory for the XY-transition, the continuous renormalization group applied to the standard φ^4 theory, the simplest typical case, and lattice gauge theory as an attempt to understand quark confinement in chromodynamics.

The next five chapters (in volume 2) cover more diverse subjects. We give an introduction to strong coupling expansions and various means of analyzing them. We then briefly introduce Monte Carlo simulations with an emphasis on the applications to gauge theories. Next we turn to the significant advances in two-dimensional conformal field theory, with a lengthy presentation of the methods as well as early results. A chapter on simple disordered systems includes sample applications of fermionic techniques with no pretence at completeness. The final chapter is devoted to random geometry and an introduction to the Polyakov model of random surfaces which illustrates the relations between string theory and statistical physics.

At the price of being perhaps a bit repetitive, we have tried in the first part to introduce the subject in an elementary fashion. It is, however, assumed that the reader has some familiarity with thermodynamics as well as with quantum field theory. We often switch from one to the other interpretation, assuming that it will not be disturbing once it is realized that the exponential of the action plays the role of the Boltzmann–Gibbs statistical weight. The last chapters cover subjects still in fast evolution.

Many important subjects could unfortunately not be covered. In random order they include dynamical critical phenomena, renormalization of σ-models or non-Abelian gauge fields except for a mention of lowest order results, topological aspects, classical solutions, instantons, monopoles, anomalies (except for the conformal one). Integrable systems are missing apart from the two-dimensional Ising model. Quantum gravity *à la Regge* is only mentioned. The list could, of course, be made much longer. Our involvement in some of the topics has certainly produced obvious biases and overemphases in certain instances. We have tried, as much as possible, to correct for these defects as well as for the numerous omissions by including at the end of each chapter a section entitled "Notes." Here we quote our sources, original articles, reviews, books and complementary material. These notes are purposely scattered through the volume, as we are sure that our quotations are very incomplete. A fair bibliography in such a large domain is beyond human capacities. Should any one feel that his or her work has not been reported or not properly mentioned, he or she is certainly right and we present our most sincere apologies. On the other hand we did not hesitate to use and sometimes follow very closely some articles or reviews which served our purpose. For instance chapter 5 is built around the definitive contributions of E. Brézin, J.-C. Le Guillou, J. Zinn-Justin and G. Parisi. Except for some further elaboration by the authors themselves, it was futile to try to improve on their work. Further examples are mentioned in the notes. It is the very nature of a survey such as this one to be inspired largely by other people's works. We hope that we did not distort or caricature them.

A book might give the illusion, especially to students, that some knowledge has become definitive and that the authors understand every part of it. This is a completely false view. No one can really fully master even his own subject, and this is luckily a source of progress. It is in the process of learning, of objecting, of finding misprints and errors, in rediscovering for oneself, that one gets the real benefits. It is very likely that, in spite of our care, many errors have crept in here and there. We welcome gladly comments and criticisms.

It was very hard to keep uniform notation throughout the text, even sometimes in the same chapter. This is a standard difficulty, especially when traditional notation in a given domain comes into conflict with those used in another one, and a compromise is necessary. We hope that this will not be a source of confusion for the reader.

We have added appendices which generally gather material in very concise form. They should be supplemented by further reading. For instance appendix C of chapter 9 is obviously insufficient to describe finite and infinite Lie algebras and their representations. This appendix is, rather, meant to induce the interested reader to study the subject further. This is also the nature of several sections where the degree of mathematical sophistication seems to increase beyond the standard background, reflecting recent trends. It was felt difficult to omit these developments but also impossible to give a proper complete introduction.

Included in small type here and there are comments, exercises and short complements ... It was felt inappropriate to develop a scholarly set of problems. In this respect the whole text can be read as a problem book.

One of the "heroes" of the whole subject of statistical physics, in one guise or another, is still to this day our old friend the Ising model. We keep a few bottles of good old French wine for the lucky person who solves it in three dimensions. It would seem appropriate to create in the theoretical physics community a prize for its solution, analogous to the one founded at the beginning of the century for the proof of Fermat's theorem. Both subjects have a similar flavour, being elementary to formulate. While it is to be presumed that the answer itself is to a large extent inessential, they motivated creative efforts (and still do) which go largely beyond the goal of solving the problem itself.

Among the many books which either overlap or amply complement the present one, the foremost are of course those in the series edited by C. Domb and M.S. Green and now J. Lebowitz, entitled *Phase transitions and critical phenomena* and published through the years by Academic Press (New York). We freely refer to this series in the notes. Let us also quote here a few others, again without pretence at exhaustivity. On the statistical side, K. Huang, *Statistical mechanics*, Wiley, New York (1963); H.E. Stanley, *In-*

troduction to phase transitions and critical phenomena, Oxford University Press (1971); S.K. Ma, *Modern theory of critical phenomena*, Benjamin, New York (1976) and *Statistical mechanics*, World Scientific, Singapore (1985); D.J. Amit, *Field theory, the renormalization group and critical phenomena*, 2nd edition, World Scientific, Singapore (1984).

Books on the path integral approach to field theory are by now numerous. Among them, the classical one is R.P. Feynman and A.R. Hibbs, *Quantum mechanics and path integrals*, McGraw Hill, New York (1965). Further aspects are covered in C. Itzykson and J.-B. Zuber, *Quantum field theory*, McGraw Hill, New York (1980); P. Ramond, *Field theory, a modern primer*, Benjamin/Cummings, Reading, Mass. (1981); J. Glimm and A. Jaffe, *Quantum physics*, Springer, New York (1981). To fill some gaps on other developments in field theory, see S. Coleman, *Aspects of symmetries*, Cambridge University Press (1985); S. Treiman, R. Jackiw, B. Zumino, E. Witten *Current algebra and anomalies*, World Scientific, Singapore (1985), and to learn about integrable systems, R. Baxter *Exactly solved models in statistical mechanics*, Academic Press, New York (1982), and M. Gaudin *La fonction d'onde de Bethe*, Masson, Paris (1983). Of course, many more books are mentioned in the notes. We are also aware that several important texts are either in preparation or will appear in the near future.

Our knowledge of English remains to this day very primitive and we apologize for our cumbersome use of a foreign language. This lack of fluency has prevented us of any attempt at humour which would have been sometimes more than welcome.

We would have never undertaken writing, were it not for the teaching opportunities that we were given by several universities and schools. One of the authors (C.I.) is grateful to his colleagues from the "Troisième cycle de Suisse Romande" in Lausanne and Geneva, from the "Département de Physique de l'Université de Louvain La Neuve" and from the "Troisième cycle de physique théorique" in Marseille for giving him the possibility to teach what became parts of this text, as well as to the staff of these institutions for providing secretarial help in preparing a French unpublished manuscript. The other author (J.M.D.) acknowledges similar opportunities afforded by the "Troisième cycle de physique théorique" in Paris.

The final and certainly most pleasant duty is, of course, to thank all those, friends, colleagues, collaborators, students and secretaries who have helped us through the years. A complete list should include all the members of the Saclay "Service de physique théorique", together with its numerous visitors and the members of the many departments, institutions and meetings which offered us generous hospitality and stimulation.

Particular thanks go to our very long time friends and colleagues R. Balian, M. Bander, M. Bauer, D. Bessis, E. Brézin, A. Cappelli, A. Coste, F. David, J. des Cloizeaux, C. De Dominicis, E. Gardner, M. Gaudin, B. Derrida, J.-M. Luck, A. Morel, P. Moussa, H. Orland, G. Parisi, Y. Pomeau, R. Lacaze, H. Saleur, R. Stora, J. Zinn-Justin, and J.-B. Zuber for friendly collaborations, endless discussions and generous advice. The final form of the manuscript owes a great deal to Dany Bunel. Let her receive here our warmest thanks for her tireless help. We are also very grateful to M. Porneuf and to the documentation staff, M. Féron, J. Delouvrier and F. Chétivaux.

Last but not least, we thank the Commissariat à l'Energie Atomique, the Institut de Recherche Fondamentale and the Service de Physique Théorique for their support.

Saclay, 1988

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DIAGRAMMATIC METHODS

This chapter is devoted to technicalities related to various expansions already encountered in volume 1, mostly those that derive from the original lattice formulation of the models, be it high or low temperature, strong coupling expansions and to some extent those arising in the guise of Feynman diagrams in the continuous framework. We shall not try to be exhaustive, but rather illustrative, relying on the reader's interest to investigate in greater depth some aspects inadequately treated. Nor shall we try to explore with great sophistication the vast domain of graph theory. There are, however, a number of common features, mostly of topological nature, which we would like to present as examples of the diversity of what looks at first sight like straightforward procedures.

7.1 General Techniques

7.1.1 Definitions and notations

Let a *labelled graph* $\mathcal{G}$ be a collection of v elements from a set of indices, and l pairs of such elements with possible duplications (i.e. multiple links). We shall also interchangeably use the word *diagram* instead of graph. This abstract object is represented by v points (vertices) and l links. Each vertex is labelled by its index.

The problem under consideration will define a set of *admissible graphs*, with a corresponding *weight* $\omega(\mathcal{G})$ (a real or complex number) according to a well-defined set of rules. We wish to find the sum of weights over all admissible graphs.

Possible constraints on the graphs may be

i) the *exclusion* constraint, preventing two vertices from carrying the same index

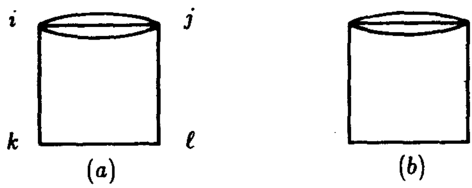

Fig. 1 (a) a labelled graph, (b) the corresponding free graph.

ii) *simplicity* when two vertices are joined by at most one link (the graph in figure 1(a) is not simple).

Take for instance the straightforward high temperature expansion of the Ising partition function

$$\begin{aligned} Z &= 2^{-N} \sum_{\{\sigma_i = \pm 1\}} \exp\left(\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j\right) \\ &= 2^{-N} \sum_{\{\sigma_i = \pm 1\}} \sum_{\langle ij \rangle} \left(1 + \sum_{n_{ij}=1}^{\infty} \frac{\beta^{n_{ij}}}{n_{ij}!} \sigma_i \sigma_j\right) \end{aligned} \tag{1}$$

Expanding the products, keeping terms with a finite power of β, and averaging over $\sigma_i = \pm 1$, leads to a straightforward high temperature series encountered in volume 1. The successive contributions are associated with graphs defined as follows. A graph has n_{ij} lines joining vertices i and j. Isolated points are not represented as vertices. Since only even powers of σ_i have a nonvanishing unit average, admissible graphs have to obey the following three rules

i) a line can only join vertices indexed by neighbouring sites, and we may think of the graph as drawn on the lattice,
ii) an even number of links are incident on a vertex,
iii) two vertices have distinct labels (the exclusion constraint).

Given an admissible graph, its weight is obtained by associating a factor β to each line, and dividing by the product $\prod_{\langle i,j \rangle} n_{ij}!$ i.e. the order of the symmetry group of the graph under permutation of equivalent links.

We can also write

$$Z = (\cosh \beta)^{Nd} \frac{1}{2^N} \sum_{\{\sigma_i = \pm 1\}} \prod_{\langle ij \rangle} \left(1 + \sigma_i \sigma_j \tanh \beta\right) \tag{2}$$

which leads for $Z/(\cosh\beta d)^N$ to a different expansion. Admissible graphs are simple with a factor $\tanh\beta$ for each link. Both series are useful in applications.

Two graphs are *isomorphic* when a one-to-one correspondence can be set among vertices and links preserving the incidence relations. The difference lies therefore in the labels of the vertices. Isomorphism leads to equivalence classes called *free graphs* and denoted G. In a pictorial representation, vertices do not carry indices anymore (fig. 1(*b*)). Conventionally, the corresponding weight $\omega(G)$ will be the average over the equivalent labelled graphs. Call number of configurations $n(G)$ the cardinal of the equivalence class, then

$$\sum_{\mathcal{G}_i \in G} \omega(\mathcal{G}_i) = n(G)\omega(G) \tag{3}$$

This definition is useful whenever the weight of a graph is independent of the labelling of its vertices. In any case, it allows one to disentangle the part $\omega(G)$ that is specific to the model, from the geometry of the lattice, which yields $n(G)$. The following two sections will treat these problems separately.

The above definitions can be extended in various ways.

i) Vertices may be of several types.
ii) Links may have to be oriented.
iii) A generalization may be envisioned, where instead of dealing with 0 and 1 dimensional simplices (vertices and links), one may be required to consider higher dimensional elements (two dimensional plaquettes in the gauge case).
iv) Indices may be compound ones, and links may have to carry indices at their extremities.

This list is of course just indicative of possible extensions.

In some applications, the computation of correlations for instance, a subset of vertices carries fixed indices. Equivalence classes of such graphs will be called *rooted graphs*.

Two vertices x and y on G are *linked* if they can be joined by a path along links of the graph xz_1, z_1z_2,..., z_ny. This provides again an equivalence relation on vertices, and the corresponding classes

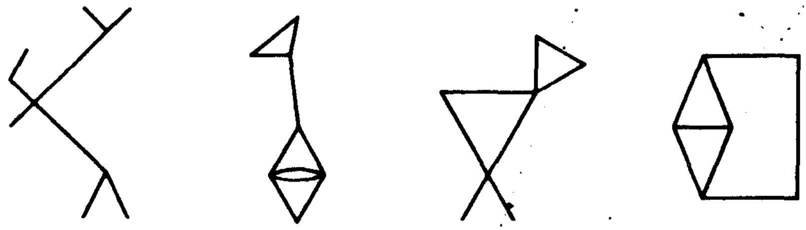

Fig. 2 (*a*) a tree (*b*) a graph with four loops (*c*) a graph with two articulation points (*d*) a multiply connected graph.

define the connected disjoint parts of the graph. A *connected* graph has a unique connected part.

A *cycle* is a closed path of n links, and n vertices, all distinct, starting and ending at the same vertex. A connected graph without cycles is a *tree* (figure 2(*a*)). The *number of loops* in a connected graph is the minimum number of links which, when removed, leave a tree (figure 2(*b*)).

An *articulation point* (figure 2(*c*)) is such that its omission, together with incident links, increases the number of connected parts. It is therefore a vertex which appears on any path linking certain pairs of vertices. In particular, on a tree, all vertices but the external ones (joined to the graph by only one link) are articulation points. A connected graph without articulation points is a *multiply connected* graph: any two vertices belong to a cycle and can therefore be linked by at least two totally distinct paths.

In terms of the following notation

v_k, number of vertices with k incident links

$v = \sum_k v_k$, total number of vertices

l, number of links

b, number of loops

c, number of connected parts

we have the relation

$$2l = \sum_k k v_k, \tag{4}$$

expressing that each link joins two vertices, thus twice the number of links is equal to the sum over vertices weighted by the number