The NUMERICAL SOLUTION of ELLIPTIC EQUATIONS

GARRETT BIRKHOFF
Harvard University

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GARRETT BIRKHOFF
Harvard University

SOCIETY for INDUSTRIAL and APPLIED MATHEMATICS
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Preface

These lecture notes are intended to survey concisely the current state of knowledge about solving elliptic boundary-value and eigenvalue problems with the help of a modern computer. For more detailed accounts of most of the relevant ideas and techniques, the reader is referred to the general references listed following this preface, to the basic references listed at the end of each lecture, and to the many research papers cited in the footnotes.

To some extent, these notes also provide a case study in *scientific computing*, by which I mean the art of utilizing physical intuition, mathematical theorems and algorithms, and modern computer technology to construct and explore realistic models of (perhaps elliptic) problems arising in the natural sciences and engineering.

As everyone knows, high-speed computers have enormously extended the range of effectively solvable partial differential equations (DE's). However, one must beware of the myth that computers have made other kinds of mathematical and scientific thinking obsolete. The kind of thinking to be avoided was charmingly expressed by C. B. Tompkins twenty years ago in his preface to [1], as follows:

> "I asked Dr. Bernstein to collect this set of existence theorems during the parlous times just after the war when it was apparent to a large and vociferous set of engineers that the electronic digital calculating machines they were then developing would replace all mathematicians and, indeed, all thinkers except the engineers building more machines.
>
> "Many of the problems presented were problems involving partial differential equations. The solution, in many cases, was to be brought about (according to the vociferous engineers) by:
>
> (1) buying a machine;
> (2) replacing the differential equation by a similar difference equation with a fine but otherwise arbitrary grid;
> (3) closing the eyes, mumbling something about truncation error and round-off error; and
> (4) pressing the button to start the machine."

The myth so wittily ridiculed by Tompkins contains a grain of truth, nevertheless. One *can* approximate almost any DE by a difference equation (ΔE) with an arbitrarily high "order of accuracy." In particular, one can approximate any *linear* DE by an approximating system of *simultaneous linear equations* which can, in principle, be solved algebraically.

In the 1950s, many techniques were invented for solving elliptic problems approximately in this way; an excellent discussion of them is contained in Forsythe and Wasow [FW, Chaps. 20–25].[1] The first five lectures below cover roughly the same material, but in more condensed style and with up-dated references. I shall devote my second lecture to a brief survey of a few facts about classical analysis which relate most specifically to elliptic DE's. My third lecture will be largely concerned with what classical analysis can say about the accuracy of difference approximations. My next two lectures will be devoted to recent developments in numerical algebra, and especially to the SOR and ADI iterative techniques for solving elliptic DE's.[2] Since 1965, the emphasis has shifted to variational methods and techniques of piecewise polynomial approximation (by "finite elements", "splines," etc.), for solving elliptic problems. My next three lectures will be primarily concerned with these methods. Note that both approximation theory and "classical" real and complex numerical algebra play essential roles in scientific computing; so does the "norm" concept of functional analysis.

However, I shall say little about classical algebra or (modern) functional analysis, because an adequate discussion of the first would lead too far afield, and because Professor Varga will cover the second in his lectures. Neither shall I say much about organization of computers, even though designers of large and frequently used "production" codes must take this into account.

The sixth lecture, which builds on the ideas introduced in the second lecture, deals with the adaptation to computers of deeper techniques from classical analysis. For these methods, which tend to apply only to special classes of problems, the book by Kantorovich and Krylov [KK] is the best reference.

The next two lectures, Lectures 7 and 8, center around recent applications of piecewise polynomial ("finite element," "Hermite," or "spline") approximations to the solution of elliptic problems having variational formulations. The last lecture reviews briefly the current status of a number of specific classes of problems, in the light of the material presented in Lectures 1–8.

Throughout, results from the following list of general references will be utilized freely. More special lists of references will also be given at the end of each chapter.

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[1] Capital letters in square brackets designate general references listed after this preface.

[2] However, I shall not discuss general techniques for computing algebraic eigenvectors and eigenvalues, because they are so masterfully discussed by Wachspress [W] and in R. S. Martin and J. H. Wilkinson, Numer. Math., 11 (1968), pp. 99–110 and G. Peters and J. H. Wilkinson, SIAM J. Numer. Anal., 7 (1970), pp. 479–492.

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THE NUMERICAL SOLUTION OF ELLIPTIC EQUATIONS

Contents

LECTURE 1

Typical Elliptic Problems

1. Two-endpoint problems. My aim in these lectures will be to describe a variety of powerful and sophisticated numerical techniques which have been used to obtain approximate solutions of elliptic differential equations and systems of equations on high-speed computers. My first lecture will be devoted to describing some typical physical problems to which these methods apply. I do this because the effective use of computers often requires physical intuition to help one decide how to formulate physical problems, which parameters are most important over which range, and whether an erratic computer output is due to physical or to numerical instability. For this reason, I shall devote my first lecture to the intuitive physical background of some of the most commonly studied elliptic problems of mathematical physics.

In elliptic problems, one is given a partial differential equation (partial DE) to be satisfied in the interior of a region R, on whose boundary ∂R additional boundary conditions are also to be satisfied by the solution. In the one-dimensional analogue of ordinary DE's, the region is an interval whose boundary consists of two endpoints. Therefore, two-endpoint problems for ordinary DE's may be regarded as boundary value problems of "elliptic" type. (By contrast, well-set initial value problems for partial DE's are typically of parabolic or hyperbolic type.)

The simplest two-endpoint problem concerns a transversely *loaded string*, in the small deflection or *linear approximation* (cf. § 8). If the string (assumed horizontal) is under a constant tension T, then the vertical deflection y induced by a load exerting a transverse force $f(x)$ per unit length satisfies the ordinary DE

$$-y'' = f(x)/T \qquad \text{(force in the } y\text{-direction)}. \tag{1}$$

If the endpoints are fixed, then the deflection satisfies also the two endpoint conditions

$$y(0) = y(c) = 0. \tag{1'}$$

The formal solution of the system (1)–(1′) is elementary: One must first find an antiderivative $g(x)$ of $f(x)$; then an antiderivative $h(x)$ of $g(x)$. Both of these are easily computed by numerical quadrature (e.g., by Simpson's rule). The general solution of (1) is then $h = h(x) + ax + b$; the boundary conditions (1′) are satisfied by some unique choice of a and b, giving the solution.

The problem of a longitudinally *loaded spring* is similar. If $p(x)$ is the stiffness of the spring, and $f(x)$ is the load per unit length, then the appropriate DE for the longitudinal deflection $y(x)$ is

$$-[p(x)y']' = f(x), \qquad p(x) > 0, \tag{2}$$

and one can again impose the fixed endpoint conditions (1′) or, more generally,

$$y(0) = y_0, \quad y(c) = y_1. \tag{2′}$$

As a third example, we consider *Sturm-Liouville systems*. These typically arise from separating out the time variable from simply harmonic solutions of time-dependent problems such as that of a vibrating string. They are defined by *homogeneous* linear DE's of the form

$$[p(x)y']' + (\lambda p(x) + q(x))\, y = 0, \tag{3}$$

in which λ is a parameter, and homogeneous linear boundary conditions of the form (1′) or, more generally,

$$\alpha_0 y(0) + \beta_0 y'(0) = \alpha_1 y(c) + \beta_1 y'(c) = 0. \tag{3′}$$

It is well known that any S-L system admits an infinite sequence $\{\lambda_i\}$ of real *eigenvalues* $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ with $\lambda_n \to \infty$, for which there are nontrivial solutions called *eigenfunctions*.

In summary, we have described above two *boundary value* problems and one *eigenvalue–eigenfunction* problem which have important higher-dimensional elliptic analogues.

2. Dirichlet and related problems. The most deeply studied elliptic boundary value (B.V.) problem is the Dirichlet problem, which can be described in physical terms as follows.

Let a homogeneous solid occupy a bounded region R in n-dimensional space, and let its boundary ∂R be kept at a specified temperature distribution $g(\mathbf{y})$ ($\mathbf{y} \in \partial R$). What will be the equilibrium temperature distribution $u(\mathbf{x})$ in the interior? Under the physically plausible (and fairly realistic) assumption that the flow ("flux") of heat at any point is proportional to the temperature gradient ∇u there, one can show that the temperature must satisfy the *Laplace equation*:

$$\nabla^2 u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \quad \text{in} \quad R \qquad (n \text{ space dimensions}). \tag{4}$$

The *Dirichlet problem* is to find a function which satisfies (4) in R and

$$u(\mathbf{y}) = g(\mathbf{y}) \quad \text{on} \quad \partial R, \tag{4a}$$

and is continuous in the closed domain $R \cup \partial R$.

The Laplace equation (4) arises in a variety of other physical contexts, often in combination with other boundary conditions. In general, a function which satisfies (4) is called *harmonic* (in R); and the study of harmonic functions (about which I shall say more in the next lecture) is called *potential theory*.

For example (see [K] or [7][1]), the Laplace equation (4) is satisfied in empty regions of space by gravitational, electrostatic and magnetostatic potentials. Thus the electrostatic potential due to a charged conductor satisfies (4) in the *exterior* of R and, in suitable units,

$$u = 1 \quad \text{on} \quad \partial R, \quad u \sim C/r \quad \text{as} \quad r \to \infty. \tag{4b}$$

The problem of solving (4) and (4b) is called the *conductor* problem, and the constant C is called the *capacity* of the conductor. Many other problems of potential theory are described in Bergman–Schiffer [1].

Likewise, the irrotational flows of an incompressible fluid studied in classical hydrodynamics [5, Chaps IV–VI] have a "velocity potential" which satisfies (4). For liquids of (nearly) constant density, this remains true under the action of gravity, a fact which makes (4) applicable also to some problems of petroleum reservoir mechanics in a homogeneous medium (soil).[2]

However, the boundary conditions which are appropriate for these applications are often quite different from those of (4a). Thus, in hydrodynamical applications, the usual boundary conditions amount to specifying *normal derivatives*,[3] or

$$\partial u/\partial n = h(\mathbf{y}) \quad \text{on} \quad \partial R. \tag{4c}$$

The problem of finding a harmonic function with given normal derivatives on the boundary is called the *Neumann problem*.

More generally, in the theory of heat conduction, it is often assumed that a solid loses heat to the surrounding air at a rate roughly proportional to the excess surface temperature (Newton's "Law of Cooling"). This leads one to try to solve (4) for the boundary conditions

$$\partial u/\partial n + ku = kg(\mathbf{y}) = h(\mathbf{y}) \quad \text{on} \quad \partial R. \tag{4d}$$

(If the conductor is cut out of sheet metal, one may look instead for functions which satisfy (4a) and the modified Helmholtz DE: $u_{xx} + u_{yy} = ku$, $k > 0$, instead of (4) inside the conductor.)

3. Membranes; source problems. Potential theory is concerned not only with harmonic functions, but also with solving the Poisson equation

$$-\nabla^2 u = f(\mathbf{x}) \tag{5}$$

in free space and in bounded domains subject to various boundary conditions. Evidently, the Poisson equation (5) is the natural generalization to $n > 1$ space dimensions of the DE (1) for the loaded string problem. Indeed, when $n = 2$, the DE (5) is satisfied by the transverse deflection $z = u(x, y)$ of a horizontal *membrane*

[1] Numbers in square brackets refer to the references listed at the end of the lecture; letters in square brackets to the list of general references after the preface.

[2] See [8]; also P. Ya. Polubarinova-Kochina, Advances in Applied Mathematics, 2 (1951), pp. 153–221, and A. E. Scheidegger, *Physics of Flow through Porous Media*, Macmillan, 1957.

[3] Here and below, $\partial/\partial n$ means *exterior* normal derivative.

(or "drumhead") under uniform lateral tension T, which supports a load of $Tf(\mathbf{x})$ per unit area.

For such a membrane, held in a fixed rigid frame, the appropriate boundary condition is

$$(5') \qquad u = 0 \quad \text{on} \quad \partial R, \quad \text{the membrane boundary.}$$

To solve (5) in R subject to the boundary condition (5′) will be one of our main concerns below.

When $n = 3$, the DE (5) is also satisfied by the gravitational potential of a continuous distribution of matter with density $4\pi f(\mathbf{x})$ per unit volume. Likewise, it is satisfied by the electrostatic potential of a continuous charge distribution having this density. These observations lead to other boundary value problems in the Poisson equation.

A more general elliptic DE is

$$(6) \qquad -\nabla \cdot [p(\mathbf{x})\nabla u] = f(\mathbf{x}).$$

It has the notable property of being *self-adjoint*, which implies that its Green's function $G(\mathbf{x}, \boldsymbol{\xi})$ (see Lecture 2, § 4) is *symmetric* in the sense that $G(\mathbf{x}, \boldsymbol{\xi}) = G(\boldsymbol{\xi}, \mathbf{x})$, and that its *eigenvalues* are *real*.

This DE is satisfied by the temperature distribution $u(\mathbf{x})$ in a solid having space-dependent thermal conductivity $p(\mathbf{x})$, in which heat is being produced at the rate of $f(\mathbf{x})/4\pi$ per unit volume and time. Since one may think of $f(\mathbf{x})$ as representing a *source* of heat, the DE (6) for suitable boundary conditions is often said to define a *source problem.* Such source problems arise typically in the analysis of diffusion phenomena. The DE (6) also arises as Darcy's Law in petroleum reservoir mechanics [8, p. 242], with p the (soil) permeability, u the pressure, and $f(\mathbf{x}) = \rho g$ constant. In practice, p may vary by orders of magnitude—like thermal and electrical conductivity.

Diffusion with convection. Another important family of elliptic DE's describes convection with diffusion. For any velocity-field $(U(x, y), V(x, y))$ with divergence $U_x + V_y = 0$, the DE

$$U\varphi_x + V\varphi_y = \alpha\nabla^2\varphi, \qquad \alpha > 0,$$

can be interpreted in this way. One should remember that, although this DE is elliptic, convection dominates diffusion in the long run, so that in many respects its solutions behave like solutions of the hyperbolic DE $U\varphi_x + V\varphi_y = 0$, but smoothed or "mollified" locally.

4. Reduced wave equation. The equation of a transversely *vibrating membrane* is

$$z_{tt} = c^2(z_{xx} + z_{yy}),$$

where $c = (T/\rho)^{1/2}$ is the wave velocity (T the tension and ρ the density per unit area of the membrane, both assumed constant). Simple harmonic (in time) oscillations of such a membrane are clearly given by setting

$$z(x, y, t) = u(x, y)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix} kt,$$

where $u(x, y)$ is a solution of the Helmholtz or *reduced wave equation*

$$\nabla^2 u + k^2 u = 0, \qquad \nabla^2 = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}, \tag{7}$$

with $n = 2$ and $x_1 = x$, $y_1 = y$. Hence, to find the possible simply harmonic vibrations of a membrane held in a rigid frame having a given contour ∂R, we must find the solutions of the Helmholtz equation (7) subject to the boundary condition

$$u = 0 \quad \text{on} \quad \partial R. \tag{7a}$$

Similarly, in three-dimensional space, let $u(\mathrm{x}) = u(x, y, z)$ be a solution of the reduced wave equation (7) with $n = 3$ in a bounded domain R with boundary ∂R, and let

$$\partial u/\partial n = 0 \quad \text{on} \quad \partial R. \tag{7b}$$

Then $p(x, y, z, t) = u(x, y, z)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix} kt$ describes the pressure variations (from ambient pressure) in a standing sound wave with frequency $ck/2\pi$ in a room (or organ pipe) having the specified (rigid) boundary ∂R.

Just as in the case of Sturm-Liouville systems (see § 1), each of the systems (7)–(7a) and (7)–(7b) has a sequence of nontrivial solutions called the eigenfunctions of the system, whose eigenvalues $\lambda_j = k_j^2$ are positive (or zero) and can be arranged in ascending order:

$$\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \cdots, \quad \lambda_n \uparrow \infty.$$

I shall discuss this "Ohm-Rayleigh principle" in the next lecture; various classical examples are worked out in textbooks on sound.[4]

Maxwell's equations. By separating out the spatial variation of simply harmonic (in time) "standing wave" solutions of Maxwell's equations for electromagnetic waves in a homogeneous medium, one is led to other solutions of the reduced wave equation. However, quantitative results about wave guides and scattering are still usually obtained by analytical methods.[5] High-speed digital computers are just beginning to be useful for solving Maxwell's equations (cf. Lecture 9).

[4] P. M. Morse, *Vibrations and Sound*, McGraw-Hill, New York, 1936.

[5] See R. E. Collin, *Field Theory of Guided Waves*, McGraw-Hill, New York, 1960, Chap. 8; L. Lewin, *Advanced Theory of Wave Guides*, Iliffe, London, 1951; N. Markuvitz, *Waveguide Handbook*, McGraw-Hill, New York, 1951.

5. Thin beams; splines. The problems discussed so far have all involved *second order* elliptic DE's. In solid mechanics, *fourth order* elliptic DE's and systems are more prevalent.

The simplest such problems refer to the small deflections of a *thin beam* or "rod" by an applied transverse "load" or force distribution. This problem was solved mathematically by the Bernoullis and Euler, who assumed that the beam or "elastica" was homogeneous, i.e., had the same physical characteristics in all cross-sections. From Hooke's law, D. Bernoulli deduced in 1706 that, in the linear or "small deflection" approximation (see § 8), the deflection of the centerline of the beam should satisfy (see [9]):

$$u^{iv}(x) = d^4u/dx^4 = f(x), \qquad a \leqq x \leqq b. \tag{8}$$

Here $f(x)$ is the quotient of the applied transverse load per unit length by the "stiffness" of the beam, whose undeflected centerline is supposed to extend along the x-axis from $x = a$ to $x = b$.

Thin beam problems can involve various sets of endpoint conditions, notably the following [CH, pp. 295–296]:

(8a) $$u''(a) = u'''(a) = u''(b) = u'''(b) = 0 \qquad \text{(free ends)}$$

(8b) $$u(a) = u''(a) = u(b) = u''(b) = 0 \qquad \text{(simply supported ends)}$$

(8c) $$u(a) = u'(a) = u(b) = u'(b) = 0 \qquad \text{(clamped ends).}$$

Regardless of the endpoint condition selected, the general solution of the DE (8) is the sum of any particular solution and some *cubic polynomial*, since the general solution of the ordinary DE $u^{iv}(x) = 0$ is a cubic polynomial. Hence, to solve any of the above two-endpoint problems for a thin beam, one can proceed as in solving (1).

Namely, one can first compute a particular solution $U(x)$ of (8) by performing four successive quadratures on $f(x)$ numerically (e.g., by Simpson's rule). One then forms

$$u(x) = U(x) + c_0 + c_1x + c_2x^2 + c_3x^3, \tag{9}$$

regarding the coefficients c_j as unknown coefficients to be determined from the four endpoint conditions.

Cubic splines. A very useful special case corresponds to "point-loads" concentrated at some sequence of points x_i:

$$\pi\colon\ a = x_0 < x_1 < x_2 < \cdots < x_{r-1} < x_r = b \quad \text{of} \quad [a, b].$$

Since, for any $c < d$,

$$u'''(d) - u'''(c) = \int_c^d u^{iv}(x) = \int_c^d f(x)\,dx,$$

a total load of w_i concentrated at x_i may be expected to produce a jump of $w_i = u'''(x_i^+) - u'''(x_i^-)$ in the third derivative of the deflection function, whose second derivative is presumably continuous. This suggests that the solutions are given by the following class of functions.

DEFINITION. A cubic spline function on $[a, b]$ with *joints* (or "knots") at the x_i, $i = 1, \cdots, r-1$, is a function $u \in C^2[a, b]$ which is expressible on each segment (x_{i-1}, x_i) by a cubic polynomial $p_i(x) = \sum_{k=0}^{3} a_{ik}x^k$.

Splines have been used by naval architects for many years to generate mechanically smooth curves which pass through (or "interpolate" to) preassigned points; as we shall see in Lectures 7 and 8, "spline functions" are also useful for computing accurate numerical solutions of elliptic DE's.

6. Plates and shells. Solid mechanics provides many challenging elliptic problems for the mathematician to solve. One of the simplest of these is provided by Kirchhoff's theory of a transversely loaded flat plate. The transverse deflection satisfies the deceptively simple-looking biharmonic equation

$$\nabla^4 u = f(x, y). \tag{10}$$

As in the one-dimensional analogue of the thin beam, one may have any of a fairly large variety of boundary conditions.

It may surprise readers to know that the DE(10) is the Euler-Lagrange DE associated with the variational condition $\delta J = 0$ for a whole family of integrals J: (10) is implied by

$$\delta\left[\int\int \{(\nabla^2 u)^2 + (1 - \nu)[u_{xx}u_{yy} - u_{xy}^2]\}\right] dx\, dy = 0 \tag{11}$$

for *any* value of the "Poisson ratio" ν (see [11]).

A parallel-loaded homogeneous plate with body force "load" potential $V(x, y)$ has stress components σ_x, σ_y and τ_{xy} most easily expressed in terms of the Airy stress function $\varphi(x, y)$:

$$\sigma_x = \varphi_{yy} + V, \quad \sigma_y = \varphi_{xx} + V, \quad \tau_{xy} = -\varphi_{xy}.$$

The conditions for static equilibrium are given by the compatibility relations

$$\nabla^4 \varphi + \frac{1 - 2\nu}{1 - \nu}\nabla^2 V = 0.$$

If V is harmonic, then φ is biharmonic.

Curvilinear elastic shells satisfy much more complicated but analogous systems of linear elliptic equations with variable coefficients.

7. Multigroup diffusion. Another important area of application for numerical methods is provided by the steady state multigroup diffusion equations of nuclear reactor theory. These constitute a *cyclic* system of DE's for source problems, in an idealized thermal reactor, typically of the form [2]:

$$\sigma_1^* \varphi_1 - \nabla \cdot ([D(x)\nabla\varphi]) = \nu\sigma_n\varphi_n \tag{12}$$

and

$$\sigma_i^* \varphi_i - \nabla \cdot [D_i(\mathbf{x}) \nabla \varphi_i] = \sigma_{i-1} \varphi_{i-1} \tag{12'}$$

for $i = 2, \cdots, n$. Here the coefficients σ_i^* and $\sigma_i \leqq \sigma_i^*$ are typically piecewise constant. These DE's are to hold in the "reactor domain" R; on the boundary ∂R of R, it is assumed that

$$l_i \partial \varphi_i / \partial n + \varphi_i = 0, \qquad i = 1, \cdots, n, \quad l_i > 0. \tag{13}$$

In the preceding DE's, the dependent variable $\varphi_i(x)$ stands for the "flux" level at $\mathbf{x}$ of neutrons of the ith velocity group and it equals $v_i N_i(\mathbf{x})$, where v_i is the (nominal) mean velocity of neutrons of the ith velocity group and $N_i(\mathbf{x})$ is the expected neutron density (population per unit volume) in the vicinity of $\mathbf{x}$; the D_i are the mean "diffusivities" of neutrons of the ith velocity group; σ_i^* and σ_i are the (macroscopic) absorption and down-scattering cross-sections; ν is the mean neutron yield per fission.

The problem is an *eigenfunction* problem; of greatest practical interest are the smallest eigenvalue ν_0 (the critical yield per fission) and the associated (positive) *critical flux distribution.*

8. Some nonlinear problems. Many important elliptic problems are *non*linear; I shall here describe only a few examples of such problems.

Probably the simplest nonlinear elliptic problem is that of a loaded string or *cable*. If we use the exact expression for the curvature $\kappa = y''/(1 + y'^2)^{3/2}$, the DE of a loaded string under a horizontal tension T_0 and vertical load $w(x)$ per unit length is

$$T_0 y'' = w(x)(1 + y'^2)^{1/2}. \tag{14}$$

The case of a catenary is $w(x) = (1 + y'^2)^{1/2}$.

Only slightly less simple is the nonlinear thin beam problem, whose DE is (in terms of arc-length s):

$$\frac{d^2\varphi}{ds^2} + R \sin \varphi = 0, \quad \frac{dy}{dx} = \tan \varphi. \tag{15}$$

Its solutions are described in detail in Love's *Elasticity*, § 262.

Plateau problem. The simplest nonlinear elliptic problem whose solution is a function of two independent variables is probably the Plateau problem [CH, vol. 2, p. 223]. In its simplest form, the problem is to minimize the *area*

$$A = \int\int (1 + z_x^2 + z_y^2)^{1/2} \, dx \, dy \tag{16}$$

of a variable surface spanned by a fixed simple closed curve γ: $x = x(\theta)$, $y = y(\theta)$, $z = z(\theta)$. Physically, this surface can be realized by a thin *soap-film* spanned by a wire loop tracing out the curve γ (a special-purpose "analogue computer").

The associated Euler-Lagrange variational equation is

$$[1 + z_y^2] z_{xx} - 2 z_x z_y z_{xy} + [1 + z_x^2] z_{yy} = 0, \tag{17}$$

which clearly reduces to the Laplace DE for a nearly flat surface, with $z_x \ll 1$, $z_y \ll 1$. This is also the DE of a surface with *mean curvature zero*.

A related nonlinear problem is that of determining (e.g., computing) the surface or surfaces with given *constant mean curvature* $(\kappa_1 + \kappa_2)/2 = M$ spanned by γ.

Nonlinear heat conduction. Actually, conductivity and specific heat are temperature-dependent, while heat transfer rates in fluids depend on the temperature gradient as well as the temperature. Therefore, more exact mathematical descriptions of heat conduction lead to nonlinear DE's.

Of these, $\nabla^2 u + e^u = 0$ has been a favorite among mathematicians because of its simplicity, but it is by no means typical. Some idea of the complexity of real heat transfer problems can be obtained by skimming through [3, Chap. 26].

9. Concluding remarks. Indeed, I want to emphasize the fact that only extremely simple or extremely important scientific and engineering problems can be profitably treated on today's computers. In my lectures, I shall emphasize such very simple problems, because their theory and computational techniques for solving them are relatively far advanced and well correlated with numerical results. In doing this, I shall try to steer a middle course between extremely general "numerical analysis without numbers," in which theorems typically refer to systems of rth order DE's in n independent variables, and "numbers without analysis," alias "experimental arithmetic."

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