Solid Mechanics Division
University of Waterloo
Waterloo, Ontario, Canada

Study No. 10

Stochastic Problems in Mechanics

Editor's Preface

This study constitutes papers presented at the Symposium on Stochastic Problems in Mechanics held at the University of Waterloo during September 24-26, 1973.

In the past, many Symposia have been held dealing with specific applications of stochastic processes such as stochastic stability, stochastic control, stochastic hydraulics, safety and reliability, etc. However, it has become increasingly apparent that many of the techniques employed in one area may find application in seemingly different areas. It was this overlap that provided the impetus for this Symposium.

Essentially six general areas were chosen to be presented, namely:

General Theory

Dynamic Systems

Structural Systems

Simulation and Numerical Methods

Fatigue

Hydrology and Fluid Mechanics

Participation was by invitation and was limited to about fifty-one persons in order to encourage active discussion. General lectures were presented by:

R.A. Heller
Virginia Polytechnic Institute

C.D. Johnson
University of Alabama

Y.K. Lin
University of Illinois

N.C. Matalas
U.S. Department of Interior Geological Survey

M. Shinozuka
Columbia University

S.K. Srinivasan
Indian Institute of Technology, Madras

In addition, eighteen papers on specific problems were presented. The contents of the Study have been rearranged from that of the Symposium in six chapters in order to reflect more closely the general areas.

The organizing committee consisted of S.T. Ariaratnam, C.V.B. Gowda, D.E. Grierson, H.H.E. Leipholz, W. Lennox, and N.C. Lind. The committee expresses its gratitude to the guest speakers and the participants for their contribution to the success of the Symposium. In addition, the committee gratefully acknowledges the assistance of Miss Jacklyn Campbell and Miss Linda Heit.

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S.T. Ariaratnam

Waterloo, 1974

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GENERAL THEORY OF STOCHASTIC PROCESSES

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INTRODUCTION

The theory of stochastic processes is now finding increasing applications to problems in physical sciences which deal with processes involving some random element in their structure. The tendency to use the results of stochastic theory in building up of models in applied science and engineering has increased considerably during the past decade and this factor has contributed very much in providing further impetus for the development of the theory itself. In addition to the deep impact on physical sciences, the theory of stochastic processes plays a vital role in the development of other realms of thought ranging from social and political sciences to ecological situations. In view of these developments, the formal probabilist is, so to say, unable to cope up with the increasing demand of these various disciplines Instances are not lacking when impatient engineers or applied scientists have successfully ventured out in search of new results in stochastic processes. The present symposium is in such a spirit.

The object of this survey is to present the general methods of stochastic processes that have an impact on the problems of physical sciences. The emphasis will be on the methods of analysis rather than on the actual solution of the problems. In the next section, we present a summary of the general methods of analysing the structural properties of stochastic processes. Then we deal with a special class of stochastic processes known as point processes. The next section is devoted to the response phenomena due to stochastic impulses. It is in this context we try to demonstrate how non-Markov processes can be analysed. The final part of this section sums up the present status of the subject from the viewpoint of applications.

2. REPRESENTATION AND STRUCTURAL PROPERTIES

2.1 Preliminary Definitions

A stochastic process is simply a collection of random variables $\{X(t)\ ,\ t \in T\}$ where T is an arbitrary set. Naturally the collection of random variables is an ordered one if T is one-dimensional. Most of our dealings will correspond only to this case and the reason for such a sever restriction will become clear presently. From classical probability theory it is clear that the stochastic process X(t) is completely specified if the joint distributional properties of the collections $\{X(t_1)\ :\ t_1 \in T\}$, $\{X(t_1),\ X(t_2)\ :\ t_1,\ t_2 \in T\}$, ... $\{X(t_1),\ X(t_2), \ldots X(t_m)\ :\ t_1,\ t_2, \ldots, t_m \in T\}$ ... are specified. Next, we note that S is the sample space of events, then by definition X(t) is a mapping from the sample space into the set of reals; in other words,

$$X(t)\ :\ S \xrightarrow{X(t)} A_t \tag{2.1}$$

where A_t is a subset (not necessarily proper) of R, the reals. This enables us to introduce the notion of state space of the stochastic process. The set S is the state space of the stochastic process $\{X(t) : t \in T\}$ where

$$S = \bigcup_{t \in T} A_t . \tag{2.2}$$

We can divide the stochastic process broadly into two classes according as

(i) S is discrete

(ii) S has the cardinality of the continuum

For a stochastic process of type (i), we can without loss of generality take S to be the set of integers. A stochastic process of type (i) is completely specified by the sequence of joint probability mass functions $\pi_n(i_1, i_2, .. i_n;\ t_1, t_2, .. t_n)$ $(n = 1,2,...)$ for all $t_j \in T$ and all integers $i_1, i_2, ... i_n$ where

$$\pi_n(i_1, i_2, ... i_n;\ t_1, t_2, ..., t_n) \tag{2.3}$$

$$= \Pr\{X(t_1) = i_1, X(t_2) = i_2, ..., X(t_n) = i_n\}.$$

On the other hand, a stochastic process of type (ii) is completely specified by the sequence of joint probability distribution functions $\rho_n(x_1, x_2, ..., x_n;\ t_1, t_2, ..., t_n)$ $(n = 1,2,...)$ for all $t_j \in T$ and $x_j \in S$ where

$$\rho_n(x_1, x_2, ..., x_n;\ t_1, t_2, ..., t_n) \tag{2.4}$$

$$= \Pr\{X(t_1) \leq x_1, X(t_2) \leq x_2, .. X(t_n) \leq x_n\}$$

or equivalently by the sequence of joint probability density function $\pi_n(x_1, x_2, ... x_n;\ t_1, t_2, ..., t_n)$ $(n = 1,2,...)$ for all $t_j \in T$ and $x_j \in S$ where

$$\pi_n(x_1, x_2, ..., x_n;\ t_1, t_2, ..., t_n) \tag{2.5}$$

$$= \lim_{\Delta_1,\Delta_2,\ldots\Delta_n\to 0} \Pr\{ x_1 \leq X(t_1) \leq x_1 + \Delta_1, \; x_2 \leq X(t_2) \leq x_2 + \Delta_2 \; x_n \leq X(t_n) \leq x_n + \Delta_n\} \Delta_1\Delta_2\ldots\Delta_n$$

We can have a further subdivision of each of the processes (i) and (ii) according as T is discrete or has the cardinality of the continuum. We shall not do so as it does not lead to any conceptual simplification.

We can use the law of multiplication of probabilities to express the joint probability density (or mass) functions (2.5) (or 2.3)) in terms of the conditional density (or mass) functions.

$$\pi_n(x_1,x_2,\ldots,x_n; t_1, t_2,\ldots,t_n) \tag{2.6}$$

$$= \pi_1(x_1,t_1)\pi_2'(x_2,t_2|x_1,t_1)\pi_3'(x_3,t_3|x_1,t_1;x_2,t_2)$$

$$\ldots\ldots \pi_n'(x_n,t_n|x_1,t_1;x_2,t_2;\ldots,x_{n-1},t_{n-1})$$

where

$$\pi_m'(x_m,t_m \; x_1,t_1; \; x_2,t_2; \; \ldots x_{m-1},t_{m-1}) \tag{2.7}$$

$$= \lim_{\Delta\to 0} \Pr\{x_m < X(t_m) < x_m + \Delta_m|X(t_1) = x_1, \ldots X(t_{m-1}) = x_{m-1}\} \qquad 2 \leq m \leq n$$

However, it is clear that the conditional density functions are equally difficult to deal with. Let us assume $t_1 < t_2 \ldots t_n$. Note that this type of ordering is possible if and only if T is one dimensional and for this reason we shall assume T be one dimensional. If we now assume

$$\pi_m'(x_m,t_m|x_1,t_1;x_2,t_2,\ldots x_{m-1},t_{m-1}) \tag{2.8}$$

$$= \pi_2'(x_m,t_m|x_{m-1},t_{m-1}) \qquad m = 2,3,\ldots$$

then it is easy to determine the functions $\pi_n(\ldots,)$ provided we are given the functions $\pi_1(.,.)$ and $\pi_2^!(.,. \quad ..)$. For in such a situation (2.6) can be written as

$$\pi_n(x_1,x_2,\ldots x_n;\ t_1,t_2,\ldots t_n) \tag{2.9}$$

$$= \pi_1(x_1,t_1)\pi_2^!(x_2,t_2|x_1,t_1)\pi_2^!(x_3,t_3|x_2,t_2)$$

$$\ldots\pi_2^!(x_n,t_n|x_{n-1},{}_t t_{n-1}).$$

A stochastic process endowed with the property expressed by (2.8) is called a Markov process. Thus a Markov process is a stochastic process in which the conditional density function of the random variable $X(t_n)$ conditional upon $X(t_1) = x_1$, $X(t_2) = x_2,\ldots$ $X(t_{n-1}) = x_{n-1}$ depends only on x_{n-1} and t_{n-1} provided $t_1 < t_{n-1}$, $t_2 < t_{n-1},\ldots t_{n-2} < t_{n-1}$, $t_1,t_2\ldots t_{n-1} < t_n$. In other words the conditional density function (2.7) depends on the value given to be assumed by the random variable at the most recent parametric value (t_{n-1} in this case) prior to t_n and is independent of the prehistory (so far as the values assumed by X(.) are concerned) prior to t_{n-1}. It is clear that such a concept is possible only if the index set T is such that its members can be well ordered like the real number system. At the outset we would like to make it clear that the possibility of a meaningful development of the theory of stochastic processes is severely restricted to the case when T is one dimensional.

A Markov stochastic process is completely specified by the two functions $\pi_1(.,.)$ and $\pi_2^!(. \quad .)$. The most interesting result is that the function $\pi_2(..,..)$ satisfies an integral equation by virtue of the Markov nature of the process. To demonstrate this let us consider any three points t_1,t_2,t_3 in the set T such that $t_1 < t_2 < t_3$. Then it is obvious that the function

$$\pi_3^!(x_2,t_2;x_3,t_3\ x_1,t_1) \tag{2.10}$$

$$= \lim_{\Delta_1,\Delta_2 \to 0} \Pr \{x_2 < X(t_2) \quad x_2 + \Delta_1,$$
$$x_3 < X(t_3) < x_3 + \Delta_2 | X(t_1) = x_1\}/\Delta_1\Delta_2$$

satisfies the relation

$$\pi_3'(x_2,t_2;x_3,t_3|x_1,t_1) = \pi_2'(x_3,t_3|x_2,t_2)\pi_2'(x_2,t_2|x_1,t_1) \tag{2.11}$$

Integrating over x_2, we obtain*

$$\pi(x_3,t_3|x_1,t_1)=\int\pi(x_3,t_3|x_2,t_2)\pi(x_2,t_2|x_1,t_1)dx_2 \tag{2.12}$$

The above integral equation is known as Chapman-Kolmogorov relation. If the state space is discrete, (2.12) takes the form

$$\pi(m,t_3|n,t_1) = \sum_{\ell} \pi(m,t_3|l,t_2)\ \pi(l,t_2|n,t_1) \tag{2.13}$$

The above equation can be written in matrix notation:

$$\Pi(t_3,t_1) = \Pi(t_3,t_2)\Pi(t_2,t_1) \tag{2.14}$$

where $\Pi(t_3,t_1)$ is the matrix with elements $\pi(m,t_3|n,t_1)$. Thus the problem becomes very simple if the state space were to be finite. For in such a case, we need solve the problem of identifying the finite matrices Π with positive elements subject to the condition expressed by (2.14) and the condition that each row of Π adds up to unity. Of course in the more general case we have to consider infinite matrices. A detailed account of such processes can be found in Kemeny and Snell (1960) and Parzen (1962).

Stochastic processes which do not satisfy the relation (2.8) form the residuary class known as non-Markov processes. Such processes are indeed difficult to deal with directly. In

*From now on we use the functional symbol π instead of π_2'.

fact non-Markov processes can be visualized as having been generated from Markov processes. To illustrate this point let us consider a mapping f defined by

$$f : X(t) \xrightarrow{f} Y(t) \tag{2.15}$$

where X(t) is a Markov process and f a many-to-one mapping. Thus when we observe the process Y(t) the points of the state space not separated by f are lumped together and hence the Markovian character is lost when we pass from X(t) to Y(t). Kendall (1964) calls such a process as "collapsed" or "lumped" Markov process. However in many practical problems Y(t) is the only process that is being observed and we may have to seek the function f and a Markov process X(t) which is the pre-image of Y(t) under f. This is by no means an easy job. Some simple examples were constructed and analysed by Cox (1955) and Mathews and Srinivasan (1956). In section 3 we shall see how non-Markovian processes can be tackled if we restrict ourselves to certain special types of stochastic processes.

2.2 *Stationarity*

There is a dichotomy in stochastic processes that is of great significance from the point of theory and applications. The probabilistic structure of X(t) may be independent of the origin of reference of t. In other words the probabilistic structure of X(t) may be invariant under arbitrary translation of t. In such a case the stochastic process is said to be stationary, the non-stationary processes forming the residuary class. There are several degrees of stationarity and we shall mention a few of the important ones, confining our attention when S has the cardinality of the continuum. A stochastic process X(t) is said to be

(i) simply stationary if

$$\pi(x,t+h) = \pi(x,t) \text{ for arbitrary } h \text{ and for all } x \in S, \quad t \in T. \tag{2.16}$$

and (ii) completely stationary if

$$\pi_m(x_1,x_2,\ldots x_m; \quad t_1+h, \; t_2+h,\ldots t_m+h) = \pi_m(x_1,x_2,\ldots x_m; \quad t_1,t_2,\ldots t_m) \tag{2.17}$$

for arbitrary h and for all $x_i \in S$, $t_i \in T$ $i = 1,2,\ldots m$ and $m = 1,2,\ldots$.

Very often the parameter t signifies time and in such a case the origin is usually chosen at $-\infty$. Sometimes we may not visualize the evolution of the process during the infinite past and in such a case stationarity would mean invariance under arbitrary but restricted time translation. Such processes are called t-homogeneous processes. In practical situations it is sufficient if we impose (2.17) for $m = 2$, in which case the process is said to be *stationary to second order*.

If $\{X(t) : t \in T\}$ is a stationary Markov process, then the Chapman-Kolmogorov relation can be written in a simpler form. First we note that $\pi(x_2,t_2|x_1,t_1)$ as defined by (2.7) is a function of t_2-t_1 only and setting $t_1 = 0$ without loss of generality, we can denote $\pi(x_2,t_2|x_1,t_1)$ simply by $\pi(x_2,t_2|x_1)$. The Chapman-Kolmogorov relation (2.12) can be written as

$$\pi(x_3,t_3|x_1) = \int\pi(x_2,t_2|x_1)\pi(x_3,t_3-t_2|x_2)dx_2 \qquad 0 < t_2 < t_3 \tag{2.18}$$

If the state space S is discrete, (2.18) takes the simple form

$$\Pi(t_3) = \Pi(t_3-t_2)\;\Pi(t_2) \qquad 0 < t_2 < t_3 \tag{2.19}$$

where $\Pi(t_3)$ is the matrix with elements $\pi(m,t_3|n)$. The above equation brings out the structure of the conditional probabilities. In the more general case where S has the cardinality of the continuum, we can show that the kernel of the integral equation (2.18) induces an operator $\rho(t)$ on the Hilbert space of bounded continuous functions over the reals given by

$$\rho(t)\ u(x) = \int \pi(y,t|x)\ u(y)\ dy \tag{2.20}$$

where $u(x)$ is a bounded continuous function of its argument over $(-\infty, +\infty)$. It is clear that (2.18) implies that

$$\rho(t_3) = \rho(t_3-t_2)\ \rho(t_2) \tag{2.21}$$

showing that the operators form a semi-group. Thus the theory of semi-group can be used to characterise the structure of the functions $\pi(.,. \ .)$. This is a very attractive and fruitful method of dealing with Markov processes. A detailed account of the general theory can be found in Loeve (1963) and Dyrkin (1965).

2.3 *Kolmogorov-Fokker-Planck Equations*

The stationary conditional probability densities $\pi(.. \ .)$ can be determined if $\pi(.,\Delta| \ .)$ is specified for infinitesimal Δ. There is a class of stochastic processes for which $\pi_2(x,\Delta|x')$ is given by

$$\pi(x,\Delta|x') = R(x|x')\Delta + o(\Delta) \qquad x \neq x' \tag{2.22}$$

The above equation implies that there is a probability mass concentration at $x = x'$, its magnitude being given by

$$1 - \Delta \int R(x|x')\ dx + o(\Delta).$$

Using Dirac delta functions, we can provide a complete description of $\pi(x,\Delta|x')$ by

$$\pi(x,\Delta|x') = \Delta R(x|x') + \delta(x-x')\{1-\Delta\int R(x\ x')dx + o(\Delta)\} \tag{2.23}$$

The above description implies that the sample paths of the stochastic process $X(t)$ consist of superpositions of step-functions. A typical sample path is shown in Figure 1. The paths are characterised by a finite number of jumps in any finite interval provided $\int R(x|x')dx < \infty$, the path remaining parallel to the t-axis between any two jumps.

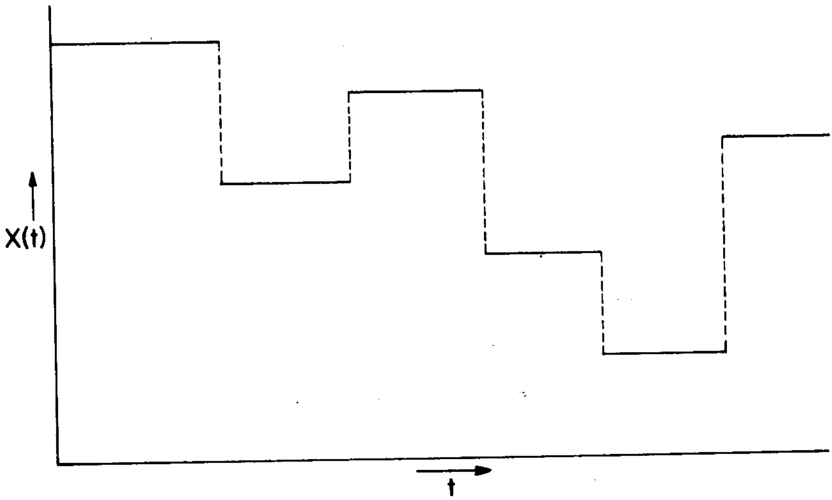

Figure 1 - A Typical Sample Path of $X(t)$

If we set $t_3 = t$, $t_2 = t_3-\Delta$ in (2.18) and make use of (2.23), we find that in the limit as Δ tends to zero,

$$\frac{\partial}{\partial t}\pi(x_3,t|x_1) = -\pi(x_3,t|x_1)\int R(x_2|x_3)dx_2 + \int \pi(x_2,t|x_1)R(x_3|x_2)dx_2 \tag{2.24}$$

an equation known as the Kolmogorov forward differential equation.

If on the other hand we set $t_3 = t$, $t_2 = \Delta$, we obtain in the limit as Δ tends to zero

$$\frac{\partial\pi(x_3,t|x_1)}{\partial t} = -\pi(x_3,t|x_1)\int R(x_2|x_1)dx_2 + \int\pi(x_3,t|x_2)\, R(x_2|x_1)dx_2 \tag{2.25}$$

which is known as the Kolmogorov backward differential equation. A detailed discussion of these equations and their solution in special cases can be found in Bharucha-Reid (1960). In passing we note that (2.24) and (2.25) are equally valid in the case when S is discrete.

There is another class of stationary stochastic processes for which the sample paths are continuous curves. To characterise this class, we do not specify $\pi(x,\Delta|x')$ directly as in (2.23). On the other hand we specify the conditional moments of $X(t+\Delta)$ conditional on $X(t) = x'$:

$$\begin{aligned} E\{[X(t+\Delta)]^n|X(t) = x'\} &= a_n(x')\Delta + o(\Delta) \qquad n = 1,2,\ldots m \\ &= o(\Delta) \qquad n > m. \end{aligned} \tag{2.26}$$

In this case, it can be shown that the Chapman-Kolmogorov relation leads to

$$\frac{\partial\pi(x,t|x')}{\partial t} = \sum_1^m \frac{(-)^n}{n!} \left(\frac{\partial}{\partial x}\right)^n a_n(x)\pi(x,t|x') \tag{2.27}$$

an equation known after Fokker and Planck. A number of physical processes involving some continuous diffusion like movement can be described by the above equation. For a detailed account, the reader is referred to Srinivasan and Vasudevan (1971).

2.4 *Representation of Random Processes*

So far we have dealt with Markov processes and outlined some methods of characterising the conditional density functions. The situation becomes slightly complex if we do not assume the Markovian property. In this section we shall attempt to characterise the processes that are stationary to second order but not necessarily Markovian. We shall also deal with a slightly more general case of complex random functions. In this general case, the function defined by (2.1) must be assumed to map S into the subset of complex numbers. Such stationary stochastic processes were introduced by Khintchine (1934); Slutsky (1937) followed it up and obtained a harmonic decomposition of random functions. A detailed account of the decomposition can be found in a survey by Moyal (1949). In this section we shall briefly mention some of the attempts by Loeve and Cramer in providing representations of random functions.

We start with some definitions. The first is continuity in quadratic mean. We say that a random function X(t) is continuous at t in quadratic mean (q.m.) if

$$E\{|x(t+h) - X(t)|^2\} \to 0 \qquad \text{as } h \to 0. \tag{2.28}$$

There is an equivalent condition for continuity in q.m. in terms of the covariance function. We shall assume without loss of generality that $E\{X(t)\} = 0$ since the expected value is a constant by virtue of stationarity. The covariance function $R(t,t')$ of the process X(t) is defined as

$$R(t,t') = E\{X(t)\,\bar{X}(t')\} \tag{2.29}$$

We shall assume that

$$E[|X(t)|^2] \qquad \text{for } \forall\, t \in T. \tag{2.30}$$