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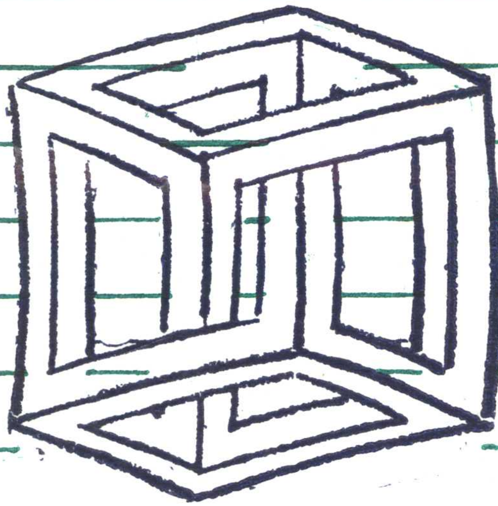
Guide To Mathematical Modelling



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EDITOR'S FOREWORD

Wide concern has been expressed in tertiary education about the difficulties experienced by students during their first year of an undergraduate course containing a substantial component of mathematics. These difficulties have a number of underlying causes, including the change of emphasis from an algorithmic approach at school to a more rigorous and abstract approach in undergraduate studies, the greater expectation of independent study, and the increased pace at which material is presented. The books in this series are intended to be sensitive to these problems.

Each book is a carefully selected, short, introductory text on a key area of the first-year syllabus; the areas are complementary and largely self-contained. Throughout, the pace of development is gentle, sympathetic and carefully motivated. Clear and detailed explanations are provided, and important concepts and results are stressed.

As mathematics is a practical subject which is best learned by doing it, rather than by watching or reading about someone else doing it, a particular effort has been made to include a plentiful supply of worked examples, together with appropriate exercises, ranging in difficulty from the straightforward to the challenging.

When one goes fellwalking, the most breathtaking views require some expenditure of effort in order to gain access to them; nevertheless, the peak is more likely to be reached if a gentle and interesting route is chosen. The mathematical peaks attainable in these books are every bit as exhilarating, the paths are as gentle as we could find, and the interest and expectation are maintained throughout to prevent the spirits from flagging on the journey.

Lancaster, 1988

David A. Towers
Consultant Editor

PREFACE

Whilst there are a number of recent texts in mathematical modelling of the 'case study' variety, these are generally of most use as source material for the teacher. This *Guide to Mathematical Modelling*, however, is intended to be read by students; so the topics treated and the order of contents have been chosen with this in mind. We have tried to address the problem of how mathematical modelling is done as well as what a mathematical model is, and so have avoided presenting just a long catalogue of completed modelling examples.

The book is essentially a first course; so the amount of prerequisite mathematics and statistics is quite modest. It is chiefly aimed at the first-year level in an undergraduate degree course in mathematical sciences, but the treatment is such that the book could be used in the second year of a school sixth form. The contents have formed the basis of the first-year modelling course for students studying for B.Sc. in Mathematics, Statistics and Computing at Thames Polytechnic and have proved a successful component in this course. The book stops short of investigating large-scale simulation models requiring software packages, but it lays valuable groundwork for subsequent study of such models.

At the outset, it is important to explain not only what modelling is, but also why it is worth doing. It is not merely a means of making the usual first-year curriculum in mathematics and statistics more lively and applicable. To accept that is to miss the point. The objective is to provide an approach to formulating and tackling problems in terms of mathematics and statistics. Eventually, when entering employment where real problems have to be dealt with, mathematicians will require additional skills to those fostered by study of conventional topics on the curriculum. The study of modelling promotes the development of these extra skills.

The book is divided into 10 chapters. Although it is not necessary to read the book strictly in chapter order, this may be preferred since there is some progression in difficulty as the subject is developed. It is vital, however, that readers try their hand at solving many of the problems posed, since modelling skills can only be learned by active participation.

Having set the scene in the opening chapter, some simple modelling problems are presented in chapter 2. These come from a variety of backgrounds, and readers should try some of the examples themselves from the problem descriptions provided. Mathematical modelling is by its nature difficult to structure, but it is useful to lay down general guidelines within which to operate when faced with new situations. To this end a general methodology is described in chapter 3.

The succeeding three chapters are particularly important for the beginner. Here the essential skills for successful modelling are developed. These are as follows.

1 Identifying the problem variables.
2 Constructing appropriate relations between these variables.
3 Taking measurements and judging the size of quantities.
4 Collecting data and deciding how to use them.
5 Estimating the values of parameters within the model that cannot be measured or calculated from data.

The backbone of the text comes in chapters 7 and 8. Chapter 7 deals with approaches to problems involving random features which demand some statistical analysis. Chapter 8 covers modelling situations which give rise to differential equations, such as are often encountered in physics and engineering.

Communication is vital for successful implementation of a mathematical model. It is necessary to explain ideas behind a model to other people, some of whom may not necessarily hold the same opinion as the modeller. It is also necessary to advise on the use of a model, often to non-specialists who need only to understand the essential points. Further, both at college and later in employment, it is often necessary to present findings verbally to a small group. These communication skills do not always come naturally; so, in chapter 9, advice is given on these matters.

Finally, in chapter 10, more demanding modelling assignments are presented. Some of the models are fully developed but others are left for the reader to process.

The content of this book complements other material usually studied in a mathematics degree course, and there is plenty of scope for further work in modelling as experience in mathematics and statistics is increased. Solving real problems by mathematical modelling is a challenging task, but it is also highly rewarding. If by working through the book readers gain confidence to take up this challenge, then the authors will be satisfied that the effort of writing the book has been worthwhile.

Woolwich 1988

D.E.
M.J.H.

CONTENTS

1 WHAT IS MODELLING?

1.1 INTRODUCTION

In industry and commerce the availability of fast and powerful computers has made it possible to 'mathematise' and 'computerise' a range of problems and activities previously unsolvable owing to their complexity. The opportunities for application of mathematics and statistics have therefore increased over the last 25 years. This means that there are more careers in industry and commerce which require a mathematical and statistical input. When the impact of complementary and related skills in computing and information technology is taken into account, there is now a very large and varied field of employment.

These opportunities can only be met if there are enough newly qualified professionals available with the right qualities to contribute. Precisely what qualities you need to develop in order to become an effective user and applier of mathematics will be described in the succeeding chapters of this book.

It is important to realise at the outset that learning to apply mathematics is a very different activity from learning mathematics. The skills needed to be successful in applying mathematics are quite different from those needed to understand concepts, to prove theorems or to solve equations. For this reason, this book is bound to appear different from a text dealing with a particular branch of mathematics. There is no theory to learn and there are only a few guiding principles. This is not to suggest, however, that mathematical modelling is an easy subject. The difficulty is not in learning and understanding the mathematics involved but in seeing where and how to apply it. There are many examples of very simple mathematics giving useful solutions to very difficult problems, although generally speaking the complexity of the problem and of the required mathematical treatment go hand in hand.

Professional modellers have to deal with a variety of real problems, and their main task is to translate each problem into a mathematical form. This is the essence of modelling, and it can involve discussions to clarify the problem, identification of problem variables, estimation, approximation and advocation of courses of action that may cost money and time.

The power industry provides many examples of how mathematical modelling is used. Problems of flow of water, electricity, gas and oil, and the necessity to match provision of these with varying demand, clearly lend themselves to mathematical treatment. The risk aspects of power provision, much publicised these days in connection with nuclear power, are also analysed by use of statistical and mathematical models.

Another activity in which mathematical modelling plays an important role is planning. Many national and local government departments depend on mathematicians to predict, for example, changes in transport, education and leisure requirements as populations shift or change in structure.

1.2 MODELS AND MODELLING

Consider the problem of optimising traffic flow near a roundabout. Unless mathematical and statistical techniques are used at the planning stage to predict the flow of traffic, the alternative is to build several differently designed roundabouts at considerable expense in order to find out which is the best. There are many situations like this where the use of mathematics provides valuable information concerning the behaviour of a system at much lower cost than the alternative 'trial-and-error' approach.

In road and traffic research laboratories, many traffic flow situations are analysed theoretically. Data are collected on the speed, size and manoeuvrability of vehicles, traffic density and junction configurations. Relations between the essential variables are then drawn up using mathematical and statistical techniques. From examination and interpretation of the results the best roundabout configuration at some particular junction can be predicted. We say that the researchers have 'built a model' of the roundabout —not a physical model but a mathematical model. The model would usually be converted into a computer simulation, which could then be used to evaluate other similar roundabout designs. Other people within the laboratory will be capable of using the model, but it is the skill of constructing the model in the first place which we wish to capture.

Enough has been said now to give you an idea of what mathematical modelling is and why it is so important. Exact definitions are not essential at this stage, but you will notice that it is easier and more useful to explain the process of modelling than to ponder on exactly what we mean by a model.

Any model (including a physical model) can be defined as a simplified representation of certain aspects of a real system. A *mathematical model* is a model created using mathematical concepts such as functions and equations. When we create mathematical models, we move from the real world into the abstract world of mathematical concepts, which is where the model is built. We then manipulate the model using mathematical techniques or computeraided numerical computation. Finally we re-enter the real world, taking with us the solution to the mathematical problem, which is then translated into a useful solution to the real problem. Note that the *start* and *end* are in the real world.

It is also important to realise at the outset that mathematical modelling is carried out in order to solve *problems*. The idea is not to produce a model which mimics a real system just for the sake of it. Any model must have a definite purpose which is clearly stated at the start. This statement may itself vary according to the point of view of the model user and there could in some cases be a clash between opposing groups regarding the particular objectives involved. For example, the effect of a new road bypass on a town centre traffic jam could be viewed differently depending on whether we side with the drivers, the pedestrians, the shopkeepers or the persons over whose land the new road will be built. It is possible that the model construction will be affected by such viewpoints, although generally it will not be the modeller's job to make a moral judgement on the issues.

It must not therefore be thought that for a particular problem there is one right and proper model. We are not in the same situation as with arithmetic or algebra, where, to each question, there is one correct answer. Many different models can be developed for tackling the same problem. (It is also true, and a remarkable demonstration of the power of mathematics, that the same abstract model can often be used for quite different physical situations.) Some models may be 'better' than others in the sense that they are more useful or more accurate, but this is not always the case. Generally the success of a model depends on how easily it can be used and how accurate are its predictions. Note also that any model will have a *limited range of validity* and should not be applied outside this range.

1.3 THE LEARNING PROCESS FOR MATHEMATICAL MODELLING

It is easy to describe real modelling problems undertaken by the professionals, but how are you to begin to develop your own expertise? As you start to read this book, you are probably reasonably confident about elementary calculus, algebra and trigonometry and perhaps also some statistics and mechanics, but constructing mathematical models is a different matter. This book aims to help you to learn how it is done. It is not necessary to attempt complicated modelling problems based on industrial applications. The 'art and craft' of model building can be learned by starting with quite commonplace situations which contain a mathematical input based only on the mathematical work done at secondary-school level. As experience and knowledge are gained both in conventional mathematics and statistics as well as in modelling, then increasingly demanding problems can be considered. The first examples need not be contrived or false, for there are plenty of simple real-life situations available for study. Within chapter 2, 10 such examples are investigated.

By the time that you have worked your way through to the end of this book, you should have gained considerable experience of mathematical models and modelling. It is important to *do* modelling yourself, to try out your own ideas and not to be afraid to risk making mistakes. Learning modelling is rather like learning to swim or to drive a car; it is no good merely reading a book on how to do it. Similarly, with modelling, it is not sufficient to read someone else's completed model. Also mathematics has perhaps acquired a reputation for being a very precise and exact subject where there is no room for debate: you are either right or wrong. Of course, it is entirely appropriate and necessary that mathematical principles are based on sound reasoning and development but, when we come to *model* some given problem, we must feel free to construct the model using whatever mathematical relationships and techniques seem appropriate, and we may well change our minds several times before we are satisfied with a particular model.

It is often important, for the best results, *not* to work on your own. In industry, it is normal for a team of people to work together on the same model, and the team may consist of engineers or economists as well as mathematicians. It should be the same for beginners at the student level; although you may read this book by yourself, we hope that most of the modelling exercises are tried amongst a group. Different people have different suggestions to make, and it is important to pool ideas.

To be a succesful mathematical modeller it is not sufficient to have expertise in the techniques of mathematics, statistics and computing. Additional skills have to be acquired, together with the following general qualities: clear thinking, a logical approach, a good feel for data, an ability to communicate and enthusiasm!

1.4 SUMMARY

1 Mathematical modelling consists of applying your mathematical skills to obtain useful answers to real problems.
2 Learning to apply mathematical skills is very different from learning mathematics itself.
3 Models are used in a very wide range of applications, some of which do not appear initially to be mathematical in nature.
4 Models often allow quick and cheap evaluation of alternatives, leading to optimal solutions which are not otherwise obvious.
5 There are no precise rules in mathematical modelling and no 'correct' answers.
6 Modelling can be learnt only by *doing*.

2 GETTING STARTED

2.1 INTRODUCTION

As we have said in chapter 1, modelling is an active pursuit which you learn best by doing yourself or in a small group. A variety of examples have been selected for 'getting started' and, while each has been developed in the text, it may be preferable to try some first directly from the problem descriptions given. On the other hand, you may be more at home after seeing how they are tackled, and in any case you may wish to check the results and 'solutions'. We must be careful here about using terms such as 'solution', since in modelling there are many cases where there is no single solution in the conventional manner. Also the term 'problem' will be used as well as 'model' and you may wonder whether there is any difference. We shall return to this point in chapter 3; in the meantime, we shall set these semantics to one side.

2.2 EXAMPLES

The 10 examples given are intended to show what modelling is about in practice. They are not all equally difficult or necessarily of equal length. The amount of mathematical technique required to support the examples is quite modest. Within some, a little support is required from more advanced topics in mathematics and statistics. (Those topics which are probably being studied alongside a first course in modelling.) Alternative approaches can be found to those given in most of the examples, and this is quite usual in mathematical modelling. As we shall see, it is also quite normal to have 'second thoughts' and to want to improve a 'solution', perhaps using more sophisticated techniques than those used here. This progression is to be encouraged, and the way that it can be done is described in the next chapter. Thus, although some of the examples can be easily extended to investigate the situations in greater depth, each has been treated initially in a straightforward manner.

Example 2.2.1: Tape

Problem description

A common problem in lagging water pipes or bandaging an injured limb is to produce a neat job without too much overlap of material. The amount of lagging required to cover a certain length of pipe will be important, but also of interest is whether there is some relation between the bandage (or tape) width, the diameter of the pipe and the angle of pitch of the tape when it is wound around so as to make a neat join. The relation between these quantities is not obvious and this is what we shall set out to find.

There are two approaches to situations such as this.

1 To try a theoretical method to deduce a formula.
2 To collect data by carrying out measurements and to attempt to obtain a formula by drawing graphs.

To proceed with either approach, the problem needs to be clearly stated and the problem variables identified. First, two assumptions will be made.

(a) All the pipes have a circular cross-section.
(b) The tape is wound so that no overlap occurs.

Now denote the tape width by W (cm), the angle of pitch by A (deg) and the circumference of a pipe by C (cm). This introduces the term 'angle of pitch', which is best explained by referring to Fig. 2.1.

Procedure

Leaving theory aside for the moment to concentrate on the practical approach, you need a sample of pipes from which measurements can be taken and a selection of tapes of different widths. A protractor is also needed to measure angles of pitch. Having made some measurements, the resulting data are then organised so that graphs can be drawn. Typical data are shown in Table 2.1. You can work with either the diameter or the circumference of a pipe.

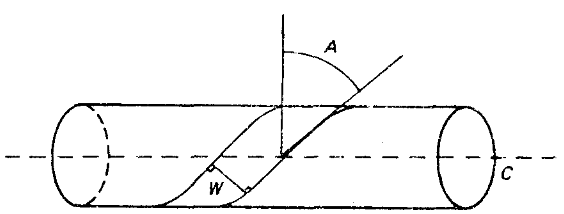

Fig. 2.1

Table 2.1

Circumference C/cm	10	10	10	10	10
Tape width W/cm	3	5	7	9	11
Angle of pitch A/deg	17	30	44	64	—
Circumference C/cm	20	20	20	20	20
Tape width W/cm	3	5	7	9	11
Angle of pitch A/deg	9	14	20	27	33
Circumference C/cm	30	30	30	30	30
Tape width W/cm	3	5	7	9	11
Angle of pitch A/deg	6	10	14	17	21

It may be difficult to deduce the required relation from graphs, especially if the measurements are not very accurate. If you are really baffled, refer to the theoretical method given at the end of the chapter, and then see whether your data fit in with the theory. Before doing this, however, think about what will happen when the tape width is either very small or very large. Carry out the resulting measurements. In particular, deduce what happens to A when

(a) W approaches zero and
(b) W becomes equal to the circumference C.

Now, as suggested just before the commencement of the examples, you may have some further ideas on the problem that you want to try out. A number of possible investigations will be listed at the end of all the examples in this chapter.

Follow-up

(a) Does it matter whether the pipe cross-sections are square or some other shape?
(b) Given a certain pipe length to lag with a particular tape supplied, how much tape do you need and what about the end effects?
(c) If the tape is p[illegible] on with overlap, what effect does this have on the results?

Example 2.2.2: Fixtures

Problem description

The tennis club captain has to arrange the order of play for the annual club mixed-doubles tournament. There is only one court available and the