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COMPUTER PROGRAMS for STRUCTURAL ANALYSIS

Computer Programs for Structural Analysis

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PREFACE

The advent of the digital computer has resulted in an extensive reorientation of structural theory from hand calculations to computer methods. Matrix formulations and programming techniques now play vital roles in the analysis of structures, and these innovations are rapidly altering the character of structural engineering. The availability of large, fast computers has stimulated many new theoretical developments, while the analysis of complex structures is becoming commonplace. With the appropriate programs, highly refined analytical models of structures subjected to various service conditions can be analyzed quickly, easily, and with a minimum of errors.

COMPUTER PROGRAMS FOR STRUCTURAL ANALYSIS contains flow charts of general-purpose programs for the analysis of framed structures. The programs are documented in a manner which is independent of a particular machine or computer language. They are arranged in a sequence of increasing difficulty commensurate with the size and complexity of the structures to be analyzed. The challenge inherent in the analysis of large structures is to conserve computer storage and run time. Factors which aid this objective are emphasized in the latter part of the book.

The stiffness method of analysis is used throughout, and the notation used in the theory is the same as that in the programs. This approach is admirably suited for programming because no engineering decisions are required to define the unknowns in the analysis. In addition, the method involves unit operations which are both systematic and localized in their effects. Stiffness matrices of framed structures are usually well conditioned, and roundoff error is rarely a problem.

In order to use this book effectively, the reader should have a mature background in the theory of structures. Matrix algebra is a prerequisite, and a prior introduction to computer programming is also desired. A previous book* dealt primarily with the fundamental theories of matrix analysis of structures, whereas the present book emphasizes the more advanced and detailed aspects of programming.

Chapter 1 summarizes the theory of the stiffness method in a form conducive to computer programming. A brief discussion of algorithmic languages and flow charts is given in Chapter 2, and a series of useful procedures for solution and inversion are developed and explained. The first general-purpose program for structural analysis appears in Chapter 3. It is capable of analyzing all types of framed structures for any number of loading conditions, using the method of matrix inversion. Chapter 4 contains a

Analysis of Framed Structures, by J. M. Gere and W. Weaver, Jr., D. Van Nostrand Company, Inc., Princeton, N. J., 1965.

second program, which takes advantage of the symmetry and band width of the stiffness matrix and obtains results by solving equilibrium equations simultaneously. The third program (Chapter 5) is oriented toward the analysis of structures of arbitrarily large size, which have been divided into substructures. Chapter 6 covers a number of supplementary programming topics that represent feasible extensions or alternatives to the basic programs. Example problems and their computer solutions are included at the ends of Chapters 3, 4, and 5. Finally, selected references for further study are listed at the end of the book.

The author is grateful to those individuals who contributed their programming talents to this project. Dr. Winfred O. Carter, Mr. Eduardo Calcaño, Capt. Joe Cannon (U. S. Army), and Mr. Robert G. Oakberg have all assisted in writing and checking the programs. Thanks are also due to Mrs. Sherry Collins and Mrs. Madelyn Hunt for typing the manuscript. Funds for the necessary computer time were provided by the School of Engineering, Stanford University.

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CONTENTS

Chapter 1

STIFFNESS METHOD OF ANALYSIS

1.1 Introduction. This chapter constitutes a summary of the *stiffness method* as applied to the analysis of framed structures. The method is presented in a form which may be readily programmed on a digital computer, and the general-purpose program in Chapter 3 follows this form.

The matrix equations of the stiffness method are general in nature, but the details of the analysis vary with the type of structure. Therefore, the basic types of framed structures to be considered are described in the next article. Equations of equilibrium for the stiffness method are formulated in Art. 1.3, and the joint stiffness matrix is defined and discussed. An outline of the steps in the analysis of a structure follows in Art. 1.4; this outline is the same as that for the computer program in Chapter 3.

Since member stiffnesses play an essential part in the analyses of all types of framed structures, this topic is treated in Art. 1.5. Next, matrices for rotation of axes are developed in Art. 1.6, primarily for the purpose of transforming member stiffness matrices from member axes to structure axes. The latter form of the member stiffness matrices is then utilized in generating the joint stiffness matrix, as shown in Art. 1.7. Loads applied to the structure are covered in Art. 1.8, and the calculation of results follows in Art. 1.9. Finally, the subject of joint displacement indexes is discussed in Art. 1.10 with the objective of segregating degrees of freedom from support restraints. The concepts described in this chapter are demonstrated by a simple numerical example, which is solved in steps paralleling the presentation of the theory.

1.2 Types of Framed Structures. All of the structures that are discussed in this book are called *framed structures*. They consist of *members* that are long in comparison with their cross-sectional dimensions. The *joints* (or *nodes*) of a framed structure are defined to be the points of intersection of the members, as well as points of support and free ends of members. It is assumed that the material of the members follows Hooke's law, that the displacements of the structure are small in comparison with its over-all dimensions, and that axial-flexural interactions are negligible. A structure which satisfies these requirements is said to be *linearly elastic*, and the principle of *superposition* can be used. The presentation in Chapters 1 through 5 is limited to structures having flexible prismatic members and inflexible connections and supports. Extensions to the theory are considered later in Chapter 6.

Every structure in this book is analyzed with respect to a set of orthogonal coordinate axes. *Actions* (forces and couples) and *displacements* (translations and rotations) are treated in component form. These components are taken to be positive when their senses correspond to the positive senses of the coordinate axes. *Loads* applied to framed structures may be concentrated forces, distributed forces, or couples.

It is convenient to divide framed structures into the following six categories: (1) continuous beams, (2) plane trusses, (3) plane frames, (4) grids, (5) space trusses, and (6) space frames. These categories are selected because each represents a class of structures having special characteristics. While the basic principles of the stiffness method are the same for all types of structures, the analyses for these six categories are sufficiently different in their details to warrant separate discussions. A general description of each type of structure follows.

Figure 1-1a shows a *continuous beam* of indefinite extent, consisting of prismatic segments rigidly connected to each other, and supported at various points along its length. A typical member i is identified in the figure

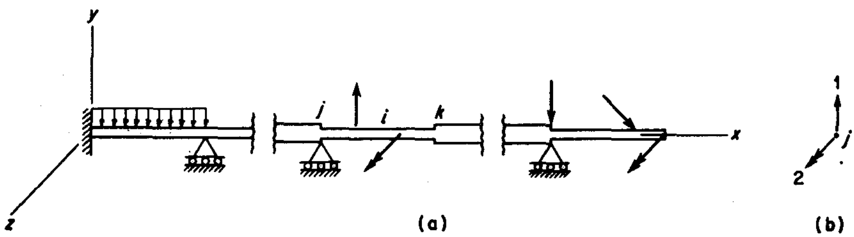

FIG. 1-1. Continuous beam

as the element framing between two joints labeled j and k. In this interval there is neither a support nor a change in cross section. Supports may be *fixed* (as at the left end) or *pinned*, or there may be *roller supports* (as at intermediate points). Concentrated and distributed applied forces are indicated in the figure by single-headed arrows. These forces lie in the x-y plane, which is assumed to contain an axis of symmetry of the cross section of the beam. Applied couples, indicated by double-headed arrows, have their moment vectors normal to the x-y plane (in the z sense). Under these conditions the beam deflects in the same plane (the *plane of bending*) and does not twist. *Internal stress resultants* may exist at any cross section of the beam and, in the general case, may include an *axial force*, a *shearing force*, and a *bending couple*.

In a continuous beam the displacements are due primarily to *flexural deformations*, and only such deformations will be considered in this chapter. The effects of *shearing deformations* can be included in an analysis, if necessary, as an extension to the theory. In either case, however, the omission of *axial deformations* means that a maximum of two possible displacements may occur at any joint. They consist of a translation in the y direction and a rotation in the z sense. Figure 1-1b shows these two types of displacements for a typical joint j. The single-headed vector labeled 1 represents the y translation, and the double-headed vector labeled 2 denotes the z rotation. This manner of depicting and numbering joint displacements will be utilized for all types of framed structures.

A portion of a *plane truss* appears in Fig. 1-2a. This type of structure is idealized as a system of members (such as member i) lying in the x-y plane and interconnected at *hinged joints* (such as joints j and k). Restraints on

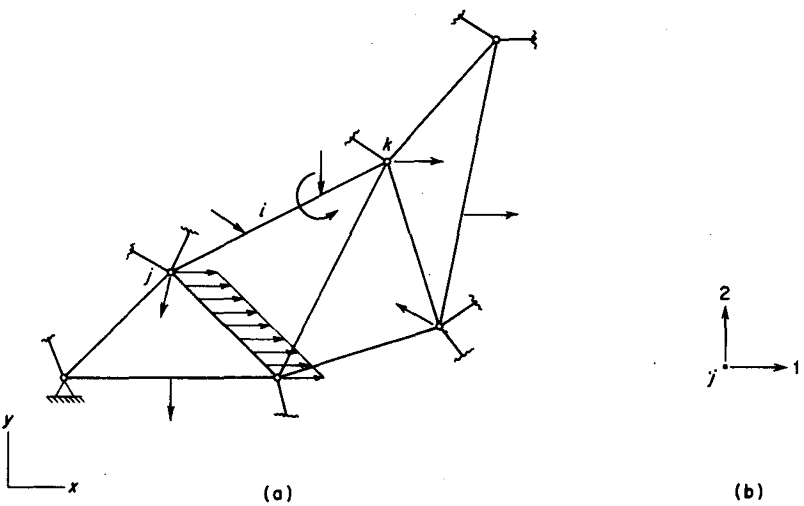

FIG. 1-2. Plane truss

the truss may be either pinned or roller supports. All applied forces are assumed to act in the plane of the structure, and applied couples have their moment vectors normal to the plane, as in the case of a beam. The loads may consist of concentrated forces applied at the joints, as well as loads that act on the members themselves. If couples are applied at the joints, they must be considered to act directly upon one or more of the members framing into that joint because of the hinge idealization. For purposes of analysis, loads applied to the members may be replaced by statically equivalent loads acting at the joints. Then the analysis of a truss subjected only to joint loads will result in axial forces of tension or compression in the members. In addition to these axial forces, there will be bending moments and shear forces in those members having loads that act directly upon them. The determination of all such stress resultants constitutes the complete analysis of the internal actions in the members of a truss. Joint translations result from axial strains in members, and these translations may be expressed conveniently by their components in the x and y directions. The two components of the translation of a typical joint j are represented by vectors 1 and 2 in Fig. 1-2b.

A *plane frame* consists of members lying in a single plane and having axes of symmetry in that plane. Figure 1-3a illustrates a part of such a

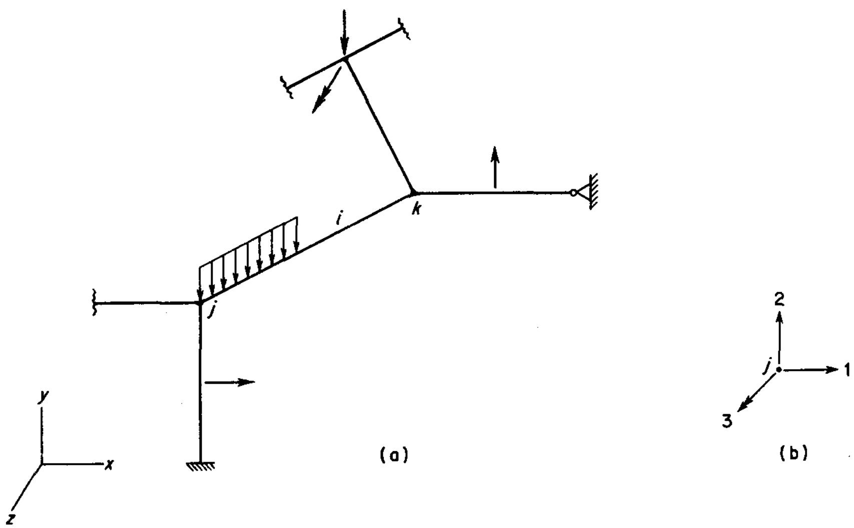

FIG. 1-3. Plane frame

structure located in the x-y plane, which is assumed to be a principal plane of bending for all of the members. A typical member i is assumed to be rigidly connected to other members at joints such as j and k. Support restraints may be fixed, pinned, or rollers. The character of the loads carried by a plane frame is the same as that for a plane truss, but couples may be considered to be applied directly to rigid joints in a frame. As in a beam, the internal stress resultants at any cross section of a plane frame member may include an axial force, a shearing force, and a bending couple. Flexural, axial, and shearing strains usually contribute to joint displacements in that order of importance, but only the first two types will be considered for the present. The three possible types of displacements at a typical joint j are indicated in Fig. 1-3b. Vectors 1 and 2 denote x and y components of the translation in the plane of the structure, and vector 3 represents the rotation in the z sense.

A *grid* is a plane structure composed of continuous members that either cross or intersect one another. In the former case the connections between members are often assumed to be hinged, whereas in the latter case the connections are assumed to be rigid. In a general analysis a grid is presumed to have rigid joints, but a grid with hinged joints may be evolved from this approach as a special case.

The coordinate axes for a grid will be taken as shown in Fig. 1-4a. The structure lies in the x-y plane, and all applied forces act parallel to the z axis. Loads in the form of couples have their moment vectors in the x-y plane.

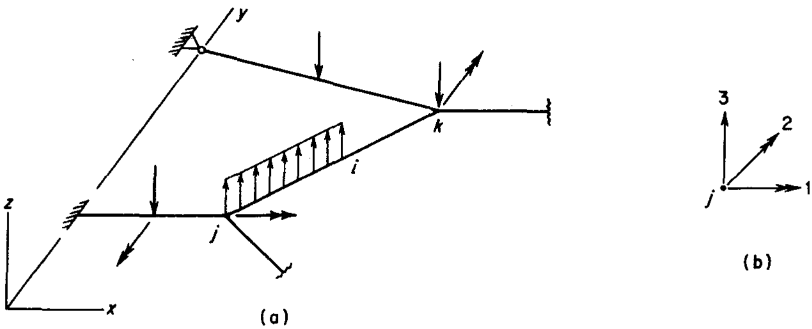

FIG. 1-4. Grid

The figure shows a typical member i framing into joints j and k. Such a member may experience torsion as well as shear and bending due to the nature of the loads. Each member is assumed to have two axes of symmetry in the cross section: one of them lying in the x-y plane and the other parallel to the z axis. This arrangement of axes guarantees that bending and torsion in the member occur independently of one another and that the member deflects in the z direction only. The significant displacements of a typical joint j are rotations in the x and y senses and a translation in the z direction, as indicated in Fig. 1-4b by vectors 1, 2, and 3, respectively.

A grid structure resembles a plane frame in several respects. All of the members and joints lie in the same plane, and the members are assumed to be rigidly connected at the joints. Flexural effects tend to predominate in the analysis of both types of structures, with the effects of torsion being secondary in the grid analysis and axial effects being secondary in the plane frame analysis. The most important difference between a plane frame and a grid is that the former is assumed to be loaded in its own plane, whereas the loads on the latter are normal to its plane. Both structures could be called plane frames, and the difference between them denoted by stating the nature of the loading system. Furthermore, if the applied loads were to have general orientations in space, the analysis of the structure could be divided into two parts. In the first part the frame would be analyzed for the components of loads in the plane of the structure, while the second part would consist of analyzing for the components of loads normal to the plane. Superposition of these two analyses would then produce the total solution of the problem. Such a structure might be considered as a special case of a space frame in which all of the members and joints lie in a common plane.

A *space truss* is similar to a plane truss, except that the members may have any directions in space. Figure 1-5a shows a portion of a space truss structure in conjunction with a set of structure axes x, y, and z. A typical

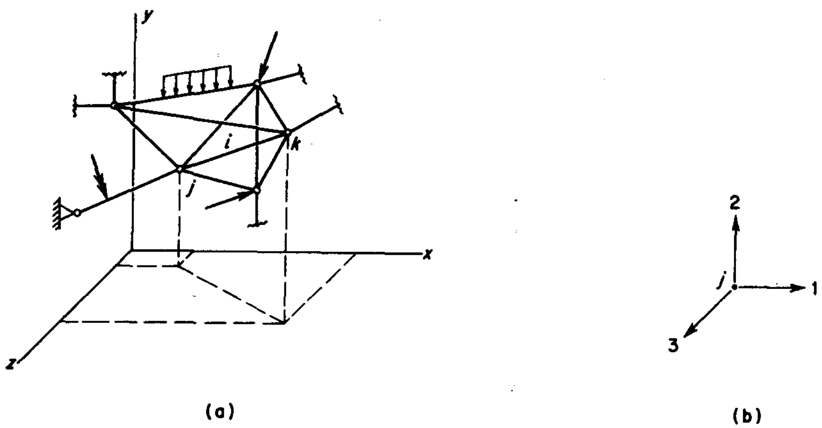

FIG. 1-5. Space truss

member i, framing into joints j and k, is indicated in the figure. All joints in a space truss are assumed to be universal hinges, and supports may be either hinges or rollers. The forces acting on such a structure may be in arbitrary directions, but any couple applied to a member must have its moment vector perpendicular to the axis of the member. The reason for this requirement is that a truss member is presumed to be incapable of transmitting a twisting moment. Internal member actions consist primarily of axial forces, but bending couples and shear forces will also exist in a member which has loads applied along its length. Because of the hinged joint idealization, rotations of the ends of the members are considered to be immaterial to the analysis. The significant joint displacements are translations due to axial strains in the members. These translations may be represented by their components in the x, y, and z directions. Figure 1-5b depicts these component translations by the vectors labeled 1, 2, and 3, respectively.

The final and most general type of structure is a *space frame*. For this type there are no restrictions on the locations of joints, orientations of members, or directions of loads. However, the cross section of each member is assumed to have two axes of symmetry, which delineate the principal planes of bending. Figure 1-6a illustrates the general character of a space frame and a typical member i within the frame. Joints such as j and k are assumed to be rigid, but restraints may be fixed, hinged, or roller supports. The individual members of a space frame may carry internal axial forces,

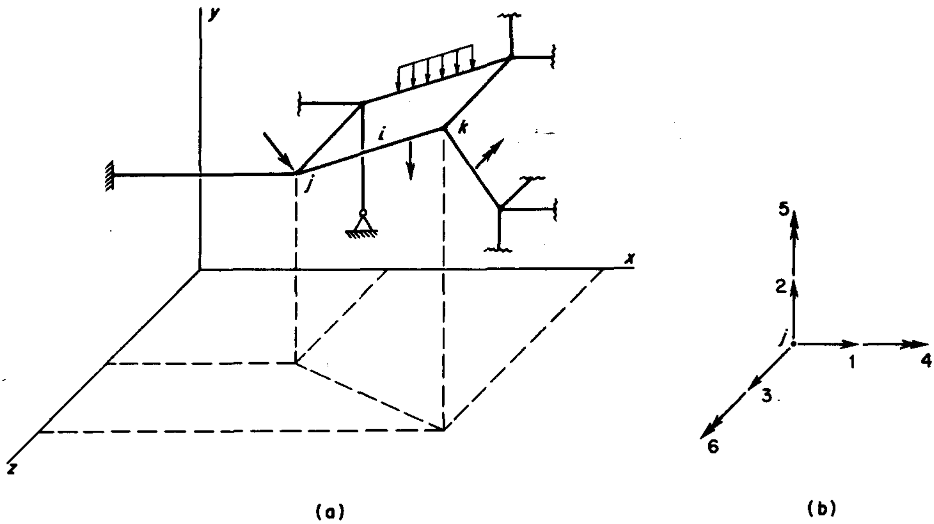

FIG. 1-6. Space frame

torsional couples, and shearing forces and bending couples in both principal directions of the cross section. Flexural, axial, and torsional strains may all be significant, and each joint that is not restrained is assumed to translate and rotate in a completely general manner in space. Thus, all possible types of joint displacements must be considered, and they are taken to be the three components of the translation and the three components of the rotation parallel to the x, y, and z axes. These six types of displacements appear in Fig. 1-6b as the three translation vectors (1, 2, and 3) and the three rotation vectors (4, 5, and 6) at the typical joint j.

An arbitrary *numbering system* for the joints and members in a framed structure will be adopted. The joints will be numbered 1 through n_j, where n_j is the total number of joints. The sequence in which the joints are numbered is immaterial, but each joint must have a number. Similarly, the members are numbered 1 through m, where m is the total number of members. Every member must have a member number i, and it will also be necessary to record the numbers j and k of the joints into which it frames. The members and joints of a continuous beam will always be numbered from left to right because of its simple geometry. However, the elements of plane and space structures may be numbered in any desired sequence, and a systematic pattern will ordinarily be adopted.

The numbering system for joint displacements follows that for numbering the joints of a structure. That is, the possible displacements at joint 1 will be numbered first, those at joint 2 numbered second, and so on. This method of identifying possible joint displacements is demonstrated for a simple plane truss structure in Fig. 1-7b. The figure shows member numbers in circles adjacent to the members, and joint numbers are indicated by numbers adjacent to the joints. The numbering system for joint displacements is represented by numbers adjacent to the arrows, which denote the positive directions of the two possible displacements at each joint. The plane truss problem shown in Fig. 1-7a will be used as an illustrative example throughout this chapter.

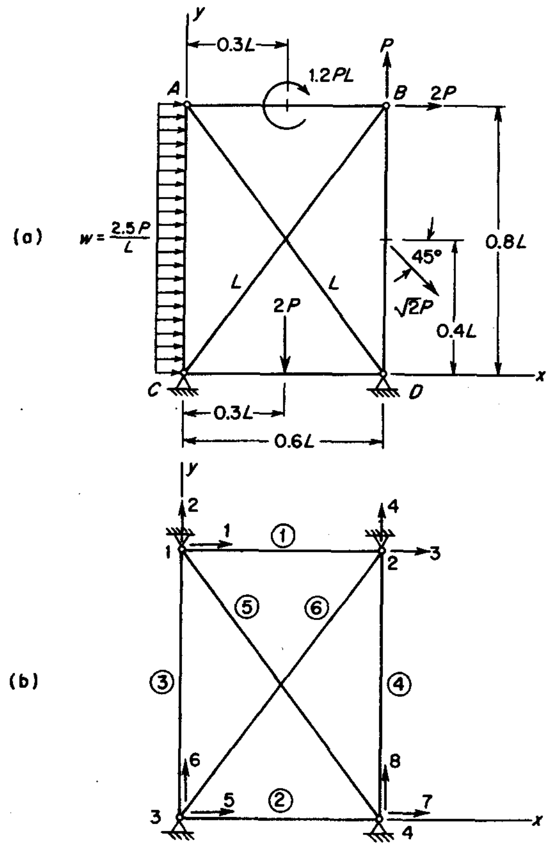

FIG. 1-7. Example (plane truss)

1.3 Equilibrium Equations — Stiffness Matrix. In the stiffness method of analysis the displacements of the joints of a structure are considered to be the basic unknowns. Equations of *joint equilibrium* may be expressed in terms of these displacements by the following matrix equation, the parts of which are explained below:

$$\mathbf{A}_J = \mathbf{A}_{JL} + \mathbf{S}_J\mathbf{D}_J \tag{1-1}$$

This formulation includes all of the joints of the structure, whether they are free to displace or are restrained by supports. Those components of joint displacements that are free to occur are called *degrees of freedom,* and the number of degrees of freedom represents the number of unknowns in the analysis. Equation (1-1) can be arranged and partitioned in an expanded form by separating the nodal degrees of freedom from the support restraints, as follows:

$$\begin{bmatrix} \mathbf{A_D} \\ \mathbf{A_R} \end{bmatrix} = \begin{bmatrix} \mathbf{A_{DL}} \\ \mathbf{A_{RL}} \end{bmatrix} + \begin{bmatrix} \mathbf{S} & \mathbf{S_{DR}} \\ \mathbf{S_{RD}} & \mathbf{S_{RR}} \end{bmatrix} \begin{bmatrix} \mathbf{D} \\ \mathbf{D_R} \end{bmatrix} \tag{1-2}$$

Thus, the symbol $\mathbf{D_J}$ represents a column vector of displacements which is divided into the unknown displacements $\mathbf{D}$ at degrees of freedom and known displacements $\mathbf{D_R}$ at support restraints. Elements of $\mathbf{D_R}$ are ordinarily zero, but they may be nonzero if support displacements are specified in a problem. Similarly, the column vector $\mathbf{A_J}$ consists of corresponding external actions $\mathbf{A_D}$ applied at free nodes, as well as support reactions $\mathbf{A_R}$ which arise at restrained nodes. The upper part of the vector $\mathbf{A_{JL}}$ contains actions $\mathbf{A_{DL}}$ at free joints due to loads applied directly to members which frame into those joints. The lower portion of $\mathbf{A_{JL}}$ is composed of actions $\mathbf{A_{RL}}$ at restrained nodes due to either loads applied directly at support restraints or loads applied to members framing into such supports. A method for treating member loads as equivalent joint loads is described in Art. 1.8.

The matrix $\mathbf{S_J}$ in Eq. (1-1) is a square, symmetric array of *stiffness influence coefficients* for actions of type $\mathbf{A_J}$ due to unit displacements of type $\mathbf{D_J}$. This matrix is called the *over-all joint stiffness matrix* (or simply *joint stiffness matrix*) because it involves all of the joints of the structure, whether they are free to displace or not. For purposes of understanding the nature of this matrix it is convenient to imagine the structure to be temporarily restrained at all joints. This conceptual aid is called the *restrained structure,* and a simple example of a completely restrained plane truss is shown in Fig. 1-7b. The elements of $\mathbf{S_J}$ for this example can be determined by inducing unit values of each nodal displacement, one at a time, and calculating the resulting *restraint actions.* This approach is not computationally efficient, but it serves to demonstrate the character of the stiffness coefficients. Figure 1-8 illustrates the effects of inducing unit amounts of the eight possible joint displacements in the restrained structure. The restraint actions indicated in the figure become elements of the joint stiffness matrix given in the following equation:

$$\mathbf{S_J} = \left[\begin{array}{cccc|cccc} S_{J11} & S_{J12} & S_{J13} & 0 & 0 & 0 & S_{J17} & S_{J18} \\ S_{J21} & S_{J22} & 0 & 0 & 0 & S_{J26} & S_{J27} & S_{J28} \\ S_{J31} & 0 & S_{J33} & S_{J34} & S_{J35} & S_{J36} & 0 & 0 \\ 0 & 0 & S_{J43} & S_{J44} & S_{J45} & S_{J46} & 0 & S_{J48} \\ \hline 0 & 0 & S_{J53} & S_{J54} & S_{J55} & S_{J56} & S_{J57} & 0 \\ 0 & S_{J62} & S_{J63} & S_{J64} & S_{J65} & S_{J66} & 0 & 0 \\ S_{J71} & S_{J72} & 0 & 0 & S_{J75} & 0 & S_{J77} & S_{J78} \\ S_{J81} & S_{J82} & 0 & S_{J84} & 0 & 0 & S_{J87} & S_{J88} \end{array}\right] \tag{a}$$

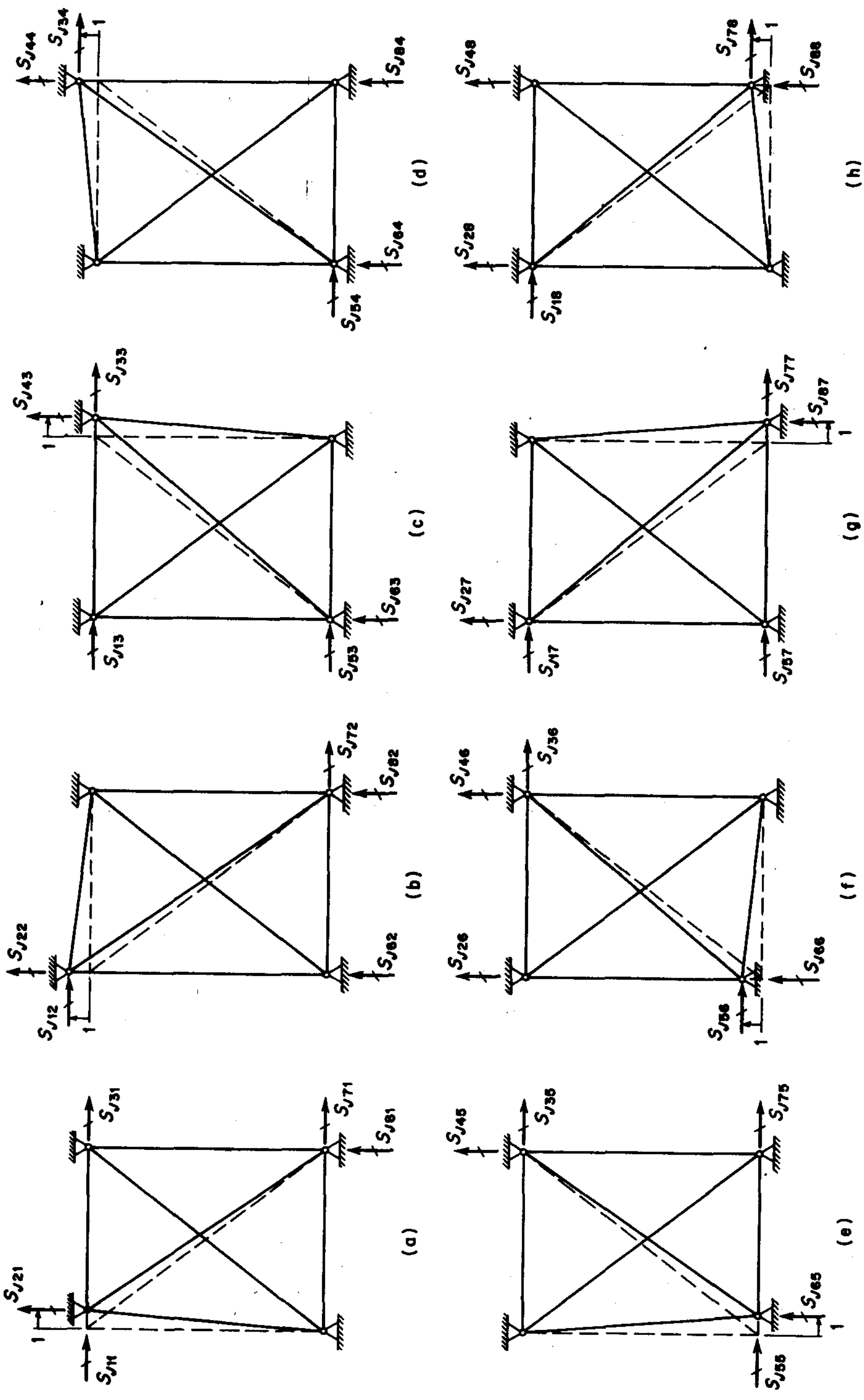

FIG. 1-8. Unit displacements for restrained truss