Discrete Mathematics for Computer Scientists

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Preface

This text is intended for use in a first course in discrete mathematics in an undergraduate computer science curriculum. The level is appropriate for a sophomore or junior course, and the number of topics and the depth of analysis can be adjusted to fit a one-term or a two-term course. This course could be taken concurrently with the student's first course in programming. It is expected that a course from this text would be preliminary to the study of the design and analysis of algorithms and data structures.

No specific mathematical background is prerequisite outside of the material ordinarily covered in most college algebra courses. In particular, a calculus background is not assumed. Moreover, the book is designed to be used by students with little or no programming experience although this would be desirable.

Our assumption about background has dictated how we have written the text in certain places. For instance, in Chapter 3, we have avoided reference to the convergence of power series by presenting the geometric series

$$\sum_{i=0}^{\infty} a^i X^i$$

as the multiplicative inverse of 1 − aX; in other words, we have considered power series from a strictly algebraic rather than the analytical viewpoint. Likewise, in Chapter 4, we avoid reference to limits when we discuss the asymptotic behavior of functions and the "big 0 notation".

The Association for Computing Machinery, CUPM, and others have recommended that a computer science curriculum include a discrete mathematics course that introduces the student to logical and algebraic structures and to combinatorial mathematics including enumeration methods and graph theory. This text is an attempt to satisfy that recommendation.

Furthermore, we expect that some of the teachers of this course will be mathematicians who are not computer scientists by profession or by training. Therefore, we have purposely suppressed writing many algorithms in computer programming language though on occasions it would have been easier to do so.

The mathematics taught to students of computer science has changed dramatically in the last fifteen to twenty years. Moreover, as the field has evolved its use of mathematics has become more sophisticated. In our view computer scientists now must have substantial training in mathematics if they are to understand their subjects well. In particular, a professional computer scientist is more than just a programmer and is called upon to design and to analyze algorithms. This requires considerable mathematical reasoning. For this reason we have included in Chapter 1 considerable detail on symbolic representation of assertions, how inferences are made, and how assertions are proved. We expect the student to return to this material from time to time as a refresher course on problem solving and methods of proof.

Much computer science is pragmatic in flavor, and therefore more like engineering than mathematics, yet other parts (for example, algorithm analysis and graph theory) are in fact, themselves mathematical topics. The trend toward more and more reliance on mathematics is likely to continue. Therefore, we make no apology for the mathematical flavor of several sections of this book. Nevertheless, we have attempted to include several motivational examples from computer science that we felt could be discussed without making presumptions about the reader's background in computer science. It is expected that subsequent courses in computer science will provide further applications of the concepts introduced in this book.

The text has evolved over a period of years and in that time we have followed different sequences in covering the topics. Thus we have written the text so that Chapter 3 can be taught at any time after Chapter 2 is covered. In particular, in a curriculum that calls for an early introduction to trees and graph theory we recommend that Chapter 3 be postponed until after Chapter 5. (Only one casual reference to the solution of the Fibonacci relation is made in Section 5.6.)

Exercises follow each section and as a general rule, the level of difficulty ranges from the routine to the moderately difficult although some proofs may present a challenge. In the early chapters we include many worked out examples and solutions to the exercises hoping to enable the student to check his work and gain confidence. Later in the book we make greater demands on the student; in particular, we expect the student to be able to make some proofs by the end of the text.

We wish to express our appreciation to several people who helped with the preparation of the manuscript. Sheila O'Connell and Pam Flowers read early versions and made several helpful suggestions while Sandy Robbins, Denise Khosrow, Lynne Pennock, Ruth Wright, Karen Serra, and Marlene Walker typed portions of the manuscript.

Finally, we want to express our love and appreciation to our families for their patience and encouragement throughout the time we were writing this book.

A Note to the Reader

In each chapter of this book, sections are numbered by chapter and then section. Thus, section number 4.2 means that it is the second section of Chapter 4. Likewise theorems, corollaries, definitions, and examples are numbered by chapter, section, and sequence so that example 4.2.7 means that the example is the seventh example in section 4.2.

The end of every theorem proof is indicated by the symbol □.

We acknowledge our intellectual debt to several authors. We have included at the end of the book a bibliography which references many, but not all, of the books that have been a great help to us. A bracket, for instance [25], means that we are referring to the article or book number 25 in the bibliography.

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Contents

1

Foundations

1.1 BASICS

One of the important tools in modern mathematics is the theory of sets. The notation, terminology, and concepts of set theory are helpful in studying any branch of mathematics. Every branch of mathematics can be considered as a study of sets of objects of one kind or another. For example, algebra is concerned with sets of numbers and operations on those sets whereas analysis deals mainly with sets of functions. The study of sets and their use in the foundations of mathematics was begun in the latter part of the nineteenth century by Georg Cantor (1845–1918). Since then, set theory has unified many seemingly disconnected ideas. It has helped to reduce many mathematical concepts to their logical foundations in an elegant and systematic way and helped to clarify the relationship between mathematics and philosophy.

What do the following have in common?

- a crowd of people,
- a herd of animals,
- a bunch of flowers, and
- a group of children.

In each case we are dealing with a collection of objects of a certain type. Rather than use a different word for each type of collection, it is convenient to denote them all by the one word "set." Thus a **set** is a collection of well-defined objects, called the **elements** of the set. The elements (or **members**) of the set are said to belong to (or be contained in) the set.

One can talk about the set of all "employees" in a corporation since an "employee" is a well-defined term. As we shall see later one can also talk about the set of "tall employees" by utilizing the concept of a **fuzzy set**.

It is important to realize that a set may itself be an element of some other set. For example, a line is a set of points; the set of all lines in the plane is a set of sets of points. In fact a set can be a set of sets of sets and so on. The theory dealing with the (abstract) sets defined in the above manner is called (**abstract** or **conventional) set theory,** in contrast to fuzzy set theory which will be introduced later.

This chapter begins with a review of set theory which includes the introduction of several important classes of sets and their properties.

In this chapter we also introduce the basic concepts of relations, functions, and lattices necessary for understanding the remainder of the material. The chapter also describes different methods of proof—including mathematical induction—and shows how to use these techniques in proving results related to the content of the text. The last section in the chapter is devoted to fuzzy sets. This section represents an area of extensive application in computer science, especially in linguistic cybernetics, approximate reasoning, and decision making in uncertain environments.

The material in Chapters 2–6 represents the applications of the concepts introduced in this chapter. Understanding these concepts and their potential applications would be sufficient mathematical preparation in these areas for most computer science students.

1.2 SETS AND OPERATIONS OF SETS

Sets will be denoted by *capital* letters $A, B, C, \ldots, X, Y, Z$. Elements will be denoted by *lower case* letters $a, b, c, \ldots, x, y, z$. The phrase "is an element of" will be denoted by the symbol $\in$. Thus we write $x \in A$ for "x is an element of A." In analogous situations, we write $x \notin A$ for "x is not an element of A."

There are five ways used to describe a set.

1. Describe a set by describing the properties of the members of the set.
2. Describe a set by listing its elements.
3. Describe a set A by its characteristic function, defined as

$$\mu_A(x) = 1 \text{ if } x \in A,$$

$$\mu_A(x) = 0 \text{ if } x \notin A,$$

for all x in U, where U is the universal set, sometimes called the "universe of discourse," or just the "universe," which is a fixed specified set describing the context for the duration.

If the discussion refers to dogs only, for example, then the universe of discourse is the class of dogs. In elementary algebra or number theory, the universe of discourse could be numbers (rational, real, complex, etc.). The universe of discourse, if any, must be explicitly stated, because the truth value of a statement depends upon it, as we shall see later.

4. Describe a set by a recursive formula. This is to give one element of the set and a rule by which the rest of the elements of the set may be found.

5. Describe a set by an operation (such as union, intersection, complement, etc.) on some sets.

Example 1.2.1. Describe the set containing all the nonnegative integers less than or equal to 5.

Let A denote the set. Set A can be described in the following ways:

1. $A = \{x \mid x$ is a nonnegative integer less than or equal to $5\}$.
2. $A = \{0,1,2,3,4,5\}$.
3. $\mu_A(x) = \begin{cases} 1 & \text{for } x = 0,1,\ldots,5, \\ 0 & \text{otherwise.} \end{cases}$
4. $A = \{x_{i+1} = x_i + 1, i = 0,1,\ldots,4, \text{ where } x_0 = 0\}$.
5. This part is left to the reader as an exercise to be completed once the operations on sets are discussed.

The use of braces and | ("such that") is a conventional notation which reads: $\{x \mid \text{property of } x\}$ means "the set of all elements x *such that* x has the given property." Note that, for a given set, not all the five ways of describing it are always possible. For example, the set of real numbers between 0 and 1 cannot be described by either listing all its elements or by a recursive formula.

In this section, we shall introduce the fundamental operations on sets and the relations among these operations. We begin with the following definitions.

Definition 1.2.1. Let A and B be two sets. A is said to be a **subset** of B if every element of A is an element of B. A is said to be a proper subset of B if A is a subset of B and there is at least one element of B which is not in A.

If A is a subset of B, we say A is contained in B. Symbolically, we write

$A \subseteq B$. If A is a proper subset of B, then we say A is strictly contained in B, denoted by $A \subset B$. The containment of sets has the following properties. Let A, B, and C be sets.

1. $A \subseteq A$.
2. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
3. If $A \subseteq B$ and $B \subset C$, then $A \subset C$.
4. If $A \subseteq B$ and $A \not\subseteq C$, then $B \not\subseteq C$, where $\not\subseteq$ means "is not contained in."

The statement $A \subseteq B$ does not rule out the possibility that $B \subseteq A$. In fact, we have both $A \subseteq B$ and $B \subseteq A$ if and only if (abbreviated iff) A and B have the same elements. Thus we define the following:

Definition 1.2.2. Two sets A and B are equal iff $A \subseteq B$ and $B \subseteq A$. We write $A = B$.

A set containing no elements is called the **empty set** or **null set,** denoted by $\varnothing$. For example, given the universal set U of all positive numbers, the set of all positive numbers x in U satisfying the equation $x + 1 = 0$ is an empty set since there are no positive numbers which can satisfy this equation. The empty set is a subset of every set. In other words, $\varnothing \subseteq A$ for every A. This is because there are no elements in $\varnothing$; therefore, every element in $\varnothing$ belongs to A. It is important to note that the sets $\varnothing$ and $\{\varnothing\}$ are very different sets. The former has no elements, whereas the latter has the unique element $\varnothing$. A set containing a single element is called a **singleton.**

We shall now describe three operations on sets; namely, complement, union, and intersection. These operations allow us to construct new sets from given sets. We shall also study the relationships among these operations.

Definition 1.2.3. Let U be the universal set and let A be any set. The **absolute complement** of A, $\overline{A}$, is defined as $\{x \mid x \notin A\}$ or, $\{x \mid x \in U \text{ and } x \notin A\}$. If A and B are sets, the **relative complement** of A with respect to B is as shown below.

$$B - A = \{x \mid x \in B \text{ and } x \notin A\}.$$

It is clear that $\overline{\varnothing} = U$, $\overline{U} = \varnothing$, and that the complement of the complement of A is equal to A.

Definition 1.2.4. Let A and B be two sets. The **union** of A and B is $A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or both}\}$. More generally, if $A_1, A_2, \ldots, A_n$ are sets, then their union is the set of all objects which belong to at least one of them, and is denoted by

$$A_1 \cup A_2 \cup \cdots \cup A_n, \text{ or by } \bigcup_{j=1}^{n} A_j.$$

Definition 1.2.5. The **intersection** of two sets A and B is $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. The intersection of n sets $A_1, A_2, \ldots, A_n$ is the set of all objects which belong to every one of them, and is denoted by

$$A_1 \cap A_2 \cap \cdots \cap A_n, \text{ or } \bigcap_{j=1}^{n} A_j.$$

Some basic properties of union and intersection of two sets are as *follows:*

	Union	Intersection
Idempotent:	$A \cup A = A$	$A \cap A = A$
Commutative:	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associative:	$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$

It should be noted that, in general,

$$(A \cup B) \cap C \neq A \cup (B \cap C).$$

Definition 1.2.6. The **symmetrical difference** of two sets A and B is $A \triangle B = \{x \mid x \in A, \text{ or } x \in B, \text{ but not both}\}$. The symmetrical difference of two sets is also called the **Boolean sum** of the two sets.

Definition 1.2.7. Two sets A and B are said to be **disjoint** if they do not have a member in common, that is to say, if $A \cap B = \varnothing$.

We can easily show the following theorems from the definitions of union, intersection, and complement.

Theorem 1.2.1. (Distributive Laws). Let A, B, and C be three sets. Then,

$$C \cap (A \cup B) = (C \cap A) \cup (C \cap B),$$
$$C \cup (A \cap B) = (C \cup A) \cap (C \cup B).$$

Theorem 1.2.2. (DeMorgan's Laws). Let A and B be two sets. Then,

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B},$$

$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}.$$

It is often helpful to use a diagram, called a Venn diagram [after John Venn (1834–1883)], to visualize the various properties of the set operations. The universal set is represented by a large rectangular area. Subsets within this universe are represented by circular areas. A summary of set operations and their Venn diagrams is given in Figure 1-1.

DeMorgan's laws can be established from the Venn diagram. If the area outside A represents $\overline{A}$ and the area outside B represents $\overline{B}$, the proof is immediate.

Let U be our universe; applying DeMorgan's laws, $A \cup B$ can be expressed as a union of disjoint sets:

$$A \cup B = \overline{(\overline{A} \cap \overline{B})} = U - (\overline{A} \cap \overline{B}) = (A \cap B) \cup (A \cap \overline{B}) \cup (\overline{A} \cap B).$$

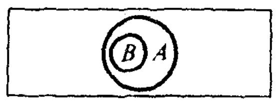
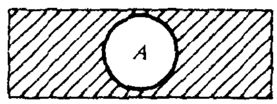
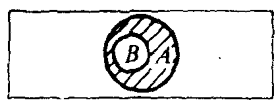
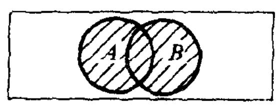
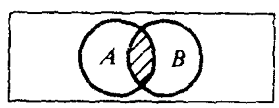
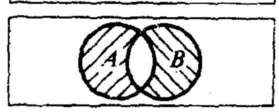
Set Operation	Symbol	Venn Diagram
Set B is contained in set A	$B \subset A$	
The absolute complement of set A	$\overline{A}$	
The relative complement of set B with respect to set A	$A - B$	
The union of sets A and B	$A \cup B$	
The intersection of sets A and B	$A \cap B$	
The symmetrical difference of sets A and B	$A \Delta B$	

Figure 1-1. Venn diagram of set operations.

Example 1.2.2.

$$
\begin{aligned}
A - (A - B) &= A - (A \cap \overline{B}) && \text{(by definition of } A - B\text{)};\\
&= A \cap (\overline{A \cap \overline{B}}) && \text{(by definition of } A - B\text{)},\\
&= A \cap (\overline{A} \cup B) && \text{(by DeMorgan)},\\
&= (A \cap \overline{A}) \cup (A \cap B) && \text{(by distributive law)},\\
&= \varnothing \cup (A \cap B) && \text{(by } A \cap \overline{A} = \varnothing\text{)},\\
&= A \cap B && \text{(by } \varnothing \cup X = X\text{)}.
\end{aligned}
$$

Clearly, the elements of a set may themselves be sets. A special class of such sets is the **power set.**

Definition 1.2.8. Let A be a given set. The **power set** of A, denoted by $\mathcal{P}(A)$, is a family of sets such that if $X \subseteq A$, then $X \in \mathcal{P}(A)$. Symbolically, $\mathcal{P}(A) = \{X \mid X \subseteq A\}$.

Example 1.2.3. Let $A = \{a,b,c\}$. The power set of A is as follows:

$$\mathcal{P}(A) = \{\{\varnothing\},\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{c,a\},\{a,b,c\}\}.$$

Exercises for Section 1.2

1. List the elements in the following sets.
   (a) The set of prime numbers less than or equal to 31.
   (b) $\{x \mid x \in \mathbb{R} \text{ and } x^2 + x - 12 = 0\}$, where $\mathbb{R}$ represents the set of real numbers.
   (c) The set of letters in the word $SUBSETS$.
2. Russell's paradox: Show that set K, such that $K = \{S \mid S \text{ is a set such that } S \notin S\}$, does not exist.
3. Prove that the empty set is unique.
4. Cantor's paradox: Show that set A, such that $A = \{S \mid S \text{ is a set}\}$, does not exist.
5. Let $U = \{1,2,3,4,5\}$, $A = \{1,5\}$, $B = \{1,2,3,4\}$, and $C = \{2,5\}$. Determine the following sets.
   (a) $A \cap \overline{B}$.
   (b) $A \cup (B \cap C)$.
   (c) $(A \cup B) \cap (A \cup C)$.
   (d) $(\overline{A \cap B}) \cup (\overline{B \cup C})$.
   (e) $\overline{A} \cup \overline{B}$.

6. Let A, B, and C be subsets of U. Prove or disprove:

$$(A \cup B) \cap (B \cup \overline{C}) \subset A \cap \overline{B}.$$

7. Use DeMorgan's laws to prove that the complement of

$$(\overline{A} \cap B) \cap (A \cup \overline{B}) \cap (A \cup C)$$

is

$$(A \cap \overline{B}) \cup (\overline{A} \cap (B \cup \overline{C})).$$

8. A_k are sets of real numbers defined as

$$A_0 = \{a \mid a \leq 1\}$$

$$A_k = \{a \mid a < 1 + 1/k\},\ k = 1, 2, \ldots.$$

Prove that

$$\bigcap_{k=1}^{\infty} A_k = A_0.$$

9. List the elements of the set $\{a/b\colon a$ and b are prime integers with $1 < a \leq 12$ and $3 < b < 9\}$.
10. Let A be a set. Define $\mathcal{P}(A)$ as the set of all subsets of A. This is called the **power set** of A. List $\mathcal{P}(A)$, where $A = \{1,2,3\}$. If $\mathcal{P}(A)$ has 256 elements, how many elements are there in A?
11. If set A has k elements, formulate a conjecture about the number of elements in $\mathcal{P}(A)$.
12. The **Cartesian product** of the sets S and T, $(S \times T)$, is the set of all ordered pairs (s,t) where $s \in S$ and $t \in T$, with $(s,t) = (u,v)$ for $u \in S$, $V \in T$, iff $s = u$ and $t = v$. Prove that $S \times T$ is not equal to $T \times S$ unless $S = T$ or either S or T is $\varnothing$.
13. Prove that $B - A$ is a subset of $\overline{A}$.
14. Prove that $B - \overline{A} = B \cap A$.
15. Prove that $A \subset B$ implies $A \cup (B - A) = B$.
16. If $A = \{0,1\}$ and $B = \{1,a\}$, determine the sets
    (a) $A \times \{1\} \times B$.
    (b) $(B \times A)^2 = (B \times A) \times (B \times A)$.