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Olli Lehto

Univalent Functions and Teichmüller Spaces

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Preface

This monograph grew out of the notes relating to the lecture courses that I gave at the University of Helsinki from 1977 to 1979, at the Eidgenössische Technische Hochschule Zürich in 1980, and at the University of Minnesota in 1982. The book presumably would never have been written without Fred Gehring's continuous encouragement. Thanks to the arrangements made by Edgar Reich and David Storvick, I was able to spend the fall term of 1982 in Minneapolis and do a good part of the writing there. Back in Finland, other commitments delayed the completion of the text.

At the final stages of preparing the manuscript, I was assisted first by Mika Seppälä and then by Jouni Luukkainen, who both had a grant from the Academy of Finland. I am greatly indebted to them for the improvements they made in the text.

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Finally, I thank the editors for accepting my text in Springer-Verlag's well-known series.

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Olli Lehto

Contents

Introduction

The theory of Teichmüller spaces studies the different conformal structures on a Riemann surface. After the introduction of quasiconformal mappings into the subject, the theory can be said to deal with classes consisting of quasiconformal mappings of a Riemann surface which are homotopic modulo conformal mappings.

It was Teichmüller who noticed the deep connection between quasiconformal mappings and function theory. He also discovered that the theory of Teichmüller spaces is intimately connected with quadratic differentials. Teichmüller ([1], [2]) proved that on a compact Riemann surface of genus greater than one, every holomorphic quadratic differential determines a quasiconformal mapping which is a unique extremal in its homotopy class in the sense that it has the smallest deviation from conformal mappings. He also showed that all extremals are obtained in this manner. It follows that the Teichmüller space of a compact Riemann surface of genus $p > 1$ is homeomorphic to the euclidean space $\mathbb{R}^{6p-6}$.

Teichmüller's proofs, often sketchy and intermingled with conjectures, were put on a firm basis by Ahlfors [1], who also introduced a more flexible definition for quasiconformal mappings. The paper of Ahlfors revived interest in Teichmüller's work and gave rise to a systematic study of the general theory of quasiconformal mappings in the plane.

Another approach to the Teichmüller theory, initiated by Bers in the early sixties, leads to quadratic differentials in an entirely different manner. This method is more general, in that it can also be applied to non-compact Riemann surfaces. The quadratic differentials are now Schwarzian derivatives of conformal extensions of quasiconformal mappings considered on the universal covering surface, the extensions being obtained by use of the Beltrami differential equation.

The development of the theory of Teichmüller spaces along these lines gives rise to several interesting problems which belong to the classical theory of univalent analytic functions. Consequently, in the early seventies a special branch of the theory of univalent functions, often studied without any connections to Riemann surfaces, began to take shape.

The interplay between the theory of univalent functions and the theory of Teichmüller spaces is the main theme of this monograph. We do give a proof of the above mentioned classical uniqueness and existence theorems of Teichmüller and discuss their consequences. But the emphasis is on the study of the repercussions of Bers's method, with attention both to univalent functions and to Teichmüller spaces. It follows that even though the topics dealt with provide an introduction to the Teichmüller theory, they leave aside many of its important aspects. Abikoff's monograph [2] and the surveys of Bers [10], [11], Earle [2], Royden [2], and Kra [2] cover material on Teichmüller spaces not treated here, and the more algebraic and differential geometric approaches, studied by Grothendieck, Bers, Earle, Thurston and many others, are not considered.

There is no clearly best way to organize our material. A lot of background knowledge is needed from the theory of quasiconformal mappings and of Riemann surfaces. A particular difficulty is caused by the fact that the interaction between univalent functions and Teichmüller spaces works in both directions.

Chapter I is devoted to an exposition of quasiconformal mappings. We have tried to collect here all the basic results that will be needed later. For detailed proofs we usually refer to the monograph Lehto–Virtanen [1]. The exceptions are cases where a brief proof can be easily presented or where we have preferred to use different arguments or, of course, where no precise reference can be given.

Chapter II deals with problems of univalent functions which have their origin in the Teichmüller theory. The leading theme is the interrelation between the Schwarzian derivative of an analytic function and the complex dilatation of its quasiconformal extension. A large fraction of the results of Chapter II comes into direct use in Chapter III concerning the universal Teichmüller space. This largest and, in many ways, simplest Teichmüller space links univalent analytic functions and general Teichmüller spaces.

A presentation of the material contained in Chapters II and III paralleling the introduction of the Teichmüller space of an arbitrary Riemann surface would perhaps have provided a better motivation for some definitions and theorems in these two chapters. But we hope that the arrangement chosen makes the theory of Teichmüller spaces of Riemann surfaces in Chapter V more transparent, as the required hard analysis has by then largely been dealt with. Also, we obtain a clear division of the book into two parts: Chapters I, II and III concern complex analysis in the plane and form an independent entity even without the rest of the book, while Chapters IV and V are related to Riemann surfaces.

The philosophy of Chapter IV on Riemann surfaces is much the same as that of Chapter I. The results needed later are formulated, and for proofs references are usually made to the standard monographs of Ahlfors–Sario [1], Lehner [1], and Springer [1]. An exception is the rather extensive treatment of holomorphic quadratic differentials, which are needed in the proof of Teichmüller's uniqueness theorem. Here we have largely utilized Strebel's monograph [6].

Finally, after all the preparations in Chapters I–IV, Teichmüller spaces of Riemann surfaces are taken up in Chapter V. We first discuss their various characterizations and, guided by the results of Chapter III, develop their general theory. After this, special attention is paid to Teichmüller spaces of compact surfaces. The torus is first treated separately and then, via the study of extremal quasiconformal mappings, compact surfaces of higher genus are discussed.

Each chapter begins with an introduction which gives a summary of its contents. The chapters are divided into sections which consist of numbered subsections. The references, such as I.2.3, are made with respect to this threefold division. In references within a chapter, the first number is omitted.

In this book, the approach to the theory of Teichmüller spaces is based on classical complex analysis. We expect the reader to be familiar with the theory of analytic functions at the level of, say, Ahlfors's standard textbook "Complex Analysis". Some basic notions of general topology, measure and integration theory and functional analysis are also used without explanations. Some acquaintance with quasiconformal mappings and Riemann surfaces would be helpful, but is not meant to be a necessary condition for comprehending the text.

CHAPTER I
Quasiconformal Mappings

Introduction to Chapter I

Quasiconformal mappings are an essential part of the contents of this book. They appear in basic definitions and theorems, and serve as a tool over and over again.

Sections 1–4 of Chapter I aim at giving the reader a quick survey of the main features of the theory of quasiconformal mappings in the plane. Complete proofs are usually omitted. For the details, an effort was made to give precise references to the literature, in most cases to the monograph Lehto–Virtanen [1].

Section 1 introduces certain conformal invariants. The Poincaré metric is repeatedly used later, and conformal modules of path families appear in the characterizations of quasiconformality.

In section 2, quasiconformality is defined by means of the maximal dilatation of a homeomorphism. Certain compactness and distortion theorems, closely related to this definition, are considered. Section 3 starts with the classical definition of quasiconformal diffeomorphisms and explains the connections between various geometric and analytic properties of quasiconformal mappings.

Section 4 is concerned with the characterization of quasiconformal mappings as homeomorphic solutions of Beltrami differential equations. Complex dilatation, a central notion throughout our presentation, is introduced, and the basic theorems about the existence, uniqueness and representation of a quasiconformal mapping with prescribed complex dilatation are discussed.

The remaining two sections are more self-contained than sections 1–4, and their contents are more clearly determined by subsequent applications. Section 5 is devoted to the now classical problem of extending a homeomorphic self-mapping of the real axis to a quasiconformal self-mapping of the half-plane. The solution is used later in several contexts.

Section 6 deals with quasidiscs. Along with the complex dilatation and the Schwarzian derivative the notion of a quasidisc is a trademark of this book. For this reason, we have given a fairly comprehensive account of their numerous geometric properties, in most cases with detailed proofs.

1. Conformal Invariants

1.1. Hyperbolic Metric

In the first three chapters of this monograph, we shall be concerned primarily with mappings whose domain and range are subsets of the plane. Unless otherwise stated, we understand by "plane" the Riemann sphere and often use the spherical metric to remove the special position of the point at infinity.

In addition to the euclidean and spherical metrics, we shall repeatedly avail ourselves of a conformally invariant hyperbolic metric. In the unit disc $D = \{z \mid |z| < 1\}$ one arrives at this metric by considering Möbius transformations $z \to w$,

$$\frac{w - w_0}{1 - \bar{w}_0 w} = e^{i\theta} \cdot \frac{z - z_0}{1 - \bar{z}_0 z}, \quad z_0, w_0 \in D,$$

which map D onto itself. By Schwarz's lemma, there are no other conformal self-mappings of D. It follows that the differential

$$\frac{|dz|}{1 - |z|^2}$$

defines a metric which is invariant under the group of conformal mappings of D onto itself.

The shortest curve in this metric joining two points z_1 and z_2 of D is the circular arc which is orthogonal to the unit circle. The hyperbolic distance between z_1 and z_2 is given by the formula

$$h(z_1, z_2) = \frac{1}{2} \log \frac{|1 - \bar{z}_1 z_2| + |z_1 - z_2|}{|1 - \bar{z}_1 z_2| - |z_1 - z_2|}. \tag{1.1}$$

The Riemann mapping theorem says that every simply connected domain A of the plane with more than one boundary point is conformally equivalent to the unit disc. Let $f: A \to D$ be a conformal mapping, and

$$\eta_A(z) = \frac{|f'(z)|}{1 - |f(z)|^2}.$$

Then the differential

$$\eta_A(z)\,|dz|$$

defines the *hyperbolic* (or *Poincaré*) *metric* of A. The function η_A, which is called the *Poincaré density* of A, is well defined, for it does not depend on the particular choice of the mapping f. In the upper half-plane, $\eta_A(z) = 1/(2\,\mathrm{Im}\,z)$. The geodesics, which are preserved under conformal mappings, are called hyperbolic segments.

The Poincaré density is monotonic with respect to the domain: If A_1 is a simply connected subdomain of A and $z \in A_1$, then

$$\eta_A(z) \leq \eta_{A_1}(z). \tag{1.2}$$

For let f and f_1 be conformal maps of A and A_1 onto the unit disc D, both vanishing at z. Then $\eta_A(z) = |f'(z)|$, $\eta_{A_1}(z) = |f_1'(z)|$, and application of Schwarz's lemma to the function $f \circ f_1^{-1}$ yields (1.2).

Similar reasoning gives an upper bound for $\eta_A(z)$ in terms of the euclidean distance $d(z, \partial A)$ from z to the boundary of A. Now we apply Schwarz's lemma to the function $\zeta \to f(z + d(z, \partial A)\zeta)$ and obtain

$$\eta_A(z) \leq \frac{1}{d(z, \partial A)}. \tag{1.3}$$

For domains A not containing ∞ we also have the lower bound

$$\eta_A(z) \geq \frac{1}{4d(z, \partial A)}. \tag{1.4}$$

This is proved by means of the *Koebe one-quarter theorem* (Nehari [2], p. 214): If f is a conformal mapping of the unit disc D with $f(0) = 0$, $f'(0) = 1$, $f(z) \neq \infty$, then $d(0, \partial f(D)) \geq 1/4$. We apply this to the function $w \to (g(w) - z)/g'(0)$, where g is a conformal mapping of D onto A with $g(0) = z$. Because $\eta_A(z) = 1/|g'(0)|$, the inequality (1.4) follows. Both estimates (1.3) and (1.4) are sharp.

There is another lower estimate for the Poincaré density which we shall need later. Let A be a simply connected domain and w_1, w_2 finite points outside A. Then

$$\eta_A(z) \geq \frac{|w_1 - w_2|}{4|z - w_1|\,|z - w_2|} \tag{1.5}$$

for every $z \in A$. To prove (1.5) we observe that $z \to f(z) = (z - w_1)/(z - w_2)$ maps A onto a domain A' which does not contain 0 or ∞. Hence, by the conformal invariance of the hyperbolic metric and by (1.4),

$$\eta_A(z) = \eta_{A'}(f(z))\,|f'(z)| \geq \frac{1}{4d(f(z), \partial A')} \cdot \frac{|w_1 - w_2|}{|z - w_2|^2}.$$

Since $d(f(z), \partial A') \leq |f(z)|$, the inequality (1.5) follows.

The hyperbolic metric can be transferred by means of conformal mappings to multiply connected plane domains with more than two boundary points and even to most Riemann surfaces. This will be explained in IV.3.6. Finally, in V.9.6 we define the hyperbolic metric on an arbitrary complex analytic manifold.

1.2. Module of a Quadrilateral

A central theme in what follows is to measure in quantitative terms the deviation of a homeomorphism from a conformal mapping. A natural way to do this is to study the change of some conformal invariant under homeomorphisms.

In 1.6 we shall exhibit a general method to produce conformal invariants which are appropriate for this purpose. Hyperbolic distance is not well suited to this objective, whereas two other special invariants, the module of a quadrilateral and that of a ring domain, have turned out to be particularly important. We shall first discuss the case of a quadrilateral.

A Jordan curve is the image of a circle under a homeomorphism of the plane. A domain whose boundary is a Jordan curve is called a Jordan domain.

Let f be a conformal mapping of a disc D onto a domain A. Suppose that A is locally connected at every point z of its boundary ∂A, i.e., that every neighborhood U of z in the plane contains a neighborhood V of z, such that $V \cap A$ is connected. Under this topological condition on A, a standard length-area argument yields the important result that f can be extended to a homeomorphism between the closures of D and A. It follows, in particular, that ∂A is a Jordan curve (Newman [1], p. 173).

Conversely, a Jordan domain is locally connected at every boundary point. We conclude that a conformal mapping of a Jordan domain onto another Jordan domain has a homeomorphic extension to the boundary, and hence to the whole plane. For such a mapping, the images of three boundary points can, modulo orientation, be prescribed arbitrarily on the boundary of the image domain. In contrast, four points on the boundary of a Jordan domain determine a conformal module, an observation we shall now make precise.

A *quadrilateral* $Q(z_1, z_2, z_3, z_4)$ is a Jordan domain and a sequence of four points z_1, z_2, z_3, z_4 on the boundary ∂Q following each other so as to determine a positive orientation of ∂Q with respect to Q. The arcs (z_1, z_2), (z_2, z_3), (z_3, z_4) and (z_4, z_1) are called the sides of the quadrilateral.

Let f be a conformal mapping of Q onto a euclidean rectangle R. If the boundary correspondence is such that f maps the four distinguished points z_1, z_2, z_3, z_4 to the vertices of R, then the mapping f is said to be *canonical*, and R is called a canonical rectangle of $Q(z_1, z_2, z_3, z_4)$. It is not difficult to prove that *every quadrilateral possesses a canonical mapping and that the canonical mapping is uniquely determined up to similarity transformations.*

The existence can be shown if we first map Q conformally onto the upper half-plane, arrange the four distinguished points in pairwise symmetric positions with respect to the origin, and finally perform a conformal mapping by means of a suitable elliptic integral. The uniqueness part follows directly from