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Sobolev Spaces

Translated from the Russian by
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Preface

The Sobolev spaces, i.e. the classes of functions with derivatives in L_p, occupy an outstanding place in analysis. During the last two decades a substantial contribution to the study of these spaces has been made; so now solutions to many important problems connected with them are known.

In the present monograph we consider various aspects of Sobolev space theory. Attention is paid mainly to the so called imbedding theorems. Such theorems, originally established by S. L. Sobolev in the 1930s, proved to be a useful tool in functional analysis and in the theory of linear and nonlinear partial differential equations.

We list some questions considered in this book.

1. What are the requirements on the measure μ for the inequality

$$(\int |u|^q d\mu)^{1/q} \leqslant C \| u \|_{S_p^l},$$

where S_p^l is the Sobolev space or its generalization, to hold?

2. What are the minimal assumptions on the domain for the Sobolev imbedding theorem to remain valid? How do these theorems vary under the degeneration of the boundaries? How does the class of admissible domains depend on additional requirements placed upon the behavior of a function near the boundary?

3. How "massive" must a subset e of the domain Ω be in order that "the Friedrichs inequality"

$$\| u \|_{L_q(\Omega)} \leqslant C \| \nabla_l u \|_{L_p(\Omega)}$$

hold for all smooth functions that vanish in a neighborhood of e?

The investigation of these and similar problems is not only of interest in its own right. By virtue of well-known general considerations it leads to conditions for the solvability of boundary value problems for elliptic equations and to theorems on the structure of the spectrum of the corresponding operators. Such applications are also included.

The selection of topics was mainly influenced by my involvement in their study, so a considerable part of the text is a report of my work in the field.

The book has no essential intersection with the monographs by S. M. Nikol'skiĭ [202], Besov, Il'in, Nikol'skiĭ [27], R. A. Adams [12], Peetre [210] and Triebel [244, 245], which were published during the last decade and are also devoted to spaces of differentiable functions.

Each of the twelve chapters of the book is divided into sections and most of the latter consist of subsections. The sections and subsections are numbered by two and three numbers, respectively (3.1 is Section 1 in Chapter 3, 1.4.3 is Subsection 3 in Section 4 in Chapter 1). Inside subsections we use an independent numbering of theorems, lemmas, propositions, corollaries, remarks, etc. If a subsection contains only one theorem or lemma then this theorem or lemma has no number. In references to the material from another section or subsection we first indicate the number of this section or subsection. For example, Theorem 1.2.1/1 means Theorem 1 in Subsection 1.2.1, (2.7/2) denotes formula (2) in Section 2.7.

The reader can obtain a general idea of the contents of the book from the Introduction. Most of the references to the literature are collected in Comments. The list of notation is given at the end of the book.

A part of this monograph was published in German in three volumes of Teubner-Texte zur Mathematik, Leipzig (Einbettungssätze für Sobolewsche Räume, Teil 1, 1979; Teil 2, 1980; Zur Theorie Sobolewscher Räume, 1981). In the present volume the material is essentially expanded and revised.

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My cordial thanks are due Leo F. Boron who kindly agreed to prepare the manuscript for publication and reduced the quantity of obscurities and slips.

It is my pleasant duty to thank Springer-Verlag for the excellent quality of this edition.

The dedication of this book to its translator and my wife Dr. T. O. Šapošnikova is a weak expression of my gratitude for her infinite patience, useful advice and constant assistance.

Leningrad, autumn 1985 V. G. Maz'ja

Contents

Introduction

In [229–231] Sobolev proved general integral inequalities for differentiable functions of several variables and applied them to a number of problems of mathematical physics. Sobolev introduced a notion of the generalized derivative and considered the Banach space $W_p^l(\Omega)$ of functions in $L_p(\Omega)$, $p \geqslant 1$, with generalized derivatives of order l summable of order p. In particular, using his theorems on the potential type integrals as well as an integral representation of functions and the properties of mollifications, Sobolev established the imbedding of $W_p^l(\Omega)$ into $L_q(\Omega)$ or $C(\Omega)$ under certain conditions on the exponents p, l, q.

Later the Sobolev theorems were generalized and refined in various ways (Kondrašov, Il'in, Gagliardo, Nirenberg etc.). In these studies the domains of functions possess the so-called cone property (each point of a domain is the vertex of a spherical cone with fixed height and angle which is situated inside the domain). Simple examples show that this condition is precise, e.g. if the boundary contains an outward "cusp" then a function in $W_p^1(\Omega)$ is not in general summable with power $pn/(n-p)$, $n > p$, contrary to the Sobolev inequality. On the other hand, looking at Fig. 1, the reader can easily see that

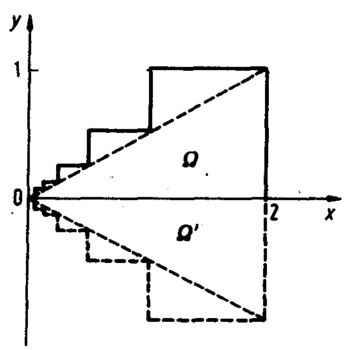

Fig. 1

the cone property is unnecessary for the imbedding $W_p^1(\Omega) \subset L_{2p/(2-p)}(\Omega)$, $2 > p$. Indeed, by unifying Ω with its mirror image, we obtain a new domain with the cone property for which the above imbedding is true by the Sobolev theorem. Consequently, the same is valid for the initial domain although it does not have the cone property.