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Linear Topological Spaces

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FOREWORD

THIS BOOK IS A STUDY OF LINEAR TOPOLOGICAL SPACES. EXPLICITLY, WE are concerned with a linear space endowed with a topology such that scalar multiplication and addition are continuous, and we seek invariants relative to the class of all topological isomorphisms. Thus, from our point of view, it is incidental that the evaluation map of a normed linear space into its second adjoint space is an isometry; it is pertinent that this map is relatively open. We study the geometry of a linear topological space for its own sake, and not as an incidental to the study of mathematical objects which are endowed with a more elaborate structure. This is not because the relation of this theory to other notions is of no importance. On the contrary, any discipline worthy of study must illuminate neighboring areas, and motivation for the study of a new concept may, in great part, lie in the clarification and simplification of more familiar notions. As it turns out, the theory of linear topological spaces provides a remarkable economy in discussion of many classical mathematical problems, so that this theory may properly be considered to be both a synthesis and an extension of older ideas.*

The text begins with an investigation of linear spaces (not endowed with a topology). The structure here is simple, and complete invariants for a space, a subspace, a linear function, and so on, are given in terms of cardinal numbers. The geometry of convex sets is the first topic which is peculiar to the theory of linear topological spaces. The fundamental propositions here (the Hahn-Banach theorem, and the relation between orderings and convex cones) yield one of the three general methods which are available for attack on linear topological space problems.

A few remarks on methodology will clarify this assertion. Our results depend primarily on convexity arguments, on compactness arguments (for example, Šmulian's compactness criterion and the Banach-Alaoglu theorem), and on category results. The chief use of scalar multiplication is made in convexity arguments; these serve to differentiate this theory from that of

* I am not enough of a scholar either to affirm or deny that all mathematics is both a synthesis and an extension of older mathematics.

topological groups. Compactness arguments—primarily applications of the Tychonoff product theorem—are important, but these follow a pattern which is routine. Category arguments are used for the most spectacular of the results of the theory. It is noteworthy that these results depend essentially on the Baire theorem for complete metric spaces and for compact spaces. There are non-trivial extensions of certain theorems (notably the Banach-Steinhaus theorem) to wider classes of spaces, but these extensions are made essentially by observing that the desired property is preserved by products, direct sums, and quotients. No form of the Baire theorem is available save for the classical cases. In this respect, the role played by completeness in the general theory is quite disappointing.

After establishing the geometric theorems on convexity we develop the elementary theory of a linear topological space in Chapter 2. With the exceptions of a few results, such as the criterion for normability, the theorems of this chapter are specializations of well-known theorems on topological groups, or even more generally, of uniform spaces. In other words, little use is made of scalar multiplication. The material is included in order that the exposition be self-contained.

A brief chapter is devoted to the fundamental category theorems. The simplicity and the power of these results justify this special treatment, although full use of the category theorems occurs later.

The fourth chapter details results on convex subsets of linear topological spaces and the closely related question of existence of continuous linear functionals, the last material being essentially a preparation for the later chapter on duality. The most powerful result of the chapter is the Krein-Milman theorem on the existence of extreme points of a compact convex set. This theorem is one of the strongest of those propositions which depend on convexity-compactness arguments, and it has far reaching consequences—for example, the existence of sufficiently many irreducible unitary representations for an arbitrary locally compact group.

The fifth chapter is devoted to a study of the duality which is the central part of the theory of linear topological spaces. The existence of a duality depends on the existence of enough continuous linear functionals—a fact which illuminates the role played by local convexity. Locally convex spaces possess a large supply of continuous linear functionals, and locally convex topologies are precisely those which may be conveniently described in terms of the adjoint space. Consequently, the duality theory, and in substance the entire theory of linear topological spaces, applies primarily to locally convex spaces. The pattern of the duality study is simple. We attempt to study a space in terms of its adjoint, and we construct part of a "dictionary" of translations of concepts defined for a space, to concepts involving the adjoint. For example, completeness of a space E is equivalent to the proposition that each hyperplane in the adjoint E^* is weak* closed whenever its intersection with every equicontinuous set A is weak* closed in A, and the topology of E is the strongest possible having E^* as the class of continuous linear functionals provided each weak* compact convex subset of E is equicontinuous. The situation is very definitely more complicated than in the case of a Banach space. Three "pleasant" properties of a space can be used to classify the type of structure. In order of increasing strength, these are: the topology for E is the strongest having E^* as adjoint (E is a Mackey space), the evaluation map of E into E^{**} is continuous (E is evaluable), and a form of the Banach-Steinhaus theorem holds for E (E is a barrelled space, or tonelé). A complete metrizable locally convex space possesses all of these properties, but an arbitrary linear topological space may fail to possess any one of them. The class of all spaces possessing any one of these useful properties is closed under formation of direct sums, products, and quotients. However, the properties are not hereditary, in the sense that a closed subspace of a space with the property may fail to have the property. Completeness, on the other hand, is preserved by the formation of direct sums and products, and obviously is hereditary, but the quotient space derived from a complete space may fail to be complete. The situation with respect to semi-reflexiveness (the evaluation map carries E onto E^{**}) is similar. Thus there is a dichotomy, and each of the useful properties of linear topological spaces follows one of two dissimilar patterns with respect to "permanence" properties.

Another type of duality suggests itself. A subset of a linear topological space is called bounded if it is absorbed by each neighborhood of 0 (that is, sufficiently large scalar multiples of any neighborhood of 0, contain the set). We may consider dually a family $\mathscr{B}$ of sets which are to be considered as bounded, and construct the family $\mathscr{U}$ of all convex circled sets which absorb members of the family $\mathscr{B}$. The family $\mathscr{U}$ defines a topology, and this scheme sets up a duality (called an internal duality) between possible topologies for E and possible families of bounded sets. This internal duality is related in a simple fashion to the dual space theory.

The chapter on duality concludes with a discussion of metrizable spaces. As might be expected, the theory of a metrizable locally convex space is more nearly perfect than that of an arbitrary space and, in fact, most of the major propositions concerning the internal structure of the dual of a Banach space hold for the adjoint of a complete metrizable space. Countability requirements are essential for many of these results. However, the structure of the second adjoint and the relation of this space to the first adjoint is still complex, and many features appear pathological compared to the classical Banach space theory.

The Appendix is intended as a bridge between the theory of linear topological spaces and that of ordered linear spaces. The elegant theorems of Kakutani characterizing Banach lattices which are of functional type, and those which are of L^1-type, are the principal results.

A final note on the preparation of this text: By fortuitous circumstance the authors were able to spend the summer of 1953 together, and a complete manuscript was prepared. We felt that this manuscript had many faults, not the least being those inferred from the old adage that a camel is a horse which was designed by a committee. Consequently, in the interest of a more uniform style, the text was revised by two of us, I. Namioka and myself. The problem lists were revised and drastically enlarged by Wendy Robertson, who, by great good fortune, was able to join in our enterprise two years ago.

J. L. K.

Berkeley, California, 1961

Note on notation: The end of each proof is marked by the symbol |||.

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Chapter 1

LINEAR SPACES

This chapter is devoted to the algebra and the geometry of linear spaces; no topology for the space is assumed. It is shown that a linear space is determined, to a linear isomorphism, by a single cardinal number, and that subspaces and linear functions can be described in equally simple terms. The structure theorems for linear spaces are valid for spaces over an arbitrary field; however, we are concerned only with real and complex linear spaces, and this restriction makes the notion of convexity meaningful. This notion is fundamental to the theory, and almost all of our results depend upon propositions about convex sets. In this chapter, after establishing connections between the geometry of convex sets and certain analytic objects, the basic separation theorems are proved. These theorems provide the foundation for linear analysis; their importance cannot be overemphasized.

1 LINEAR SPACES

Each linear space is characterized, to a linear isomorphism, by a cardinal number called its dimension. A subspace is characterized by its dimension and its co-dimension. After these results have been established, certain technical propositions on linear functions are proved (for example, the induced map theorem, and the theorem giving the relation between the linear functionals on a complex linear space and the functionals on its real restriction). The section ends with a number of definitions, each giving a method of constructing new linear spaces from old.

A **real (complex) linear space** (also called a **vector space** or a **linear space over the real** (respectively, **complex**) **field**) is a non-void set E and two operations called **addition** and **scalar multiplication.** Addition is an operation $\oplus$ which satisfies the following axioms:

(i) For every pair of elements x and y in E, $x \oplus y$, called the **sum** of x and y, is an element of E;
(ii) addition is commutative: $x \oplus y = y \oplus x$;
(iii) addition is associative: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(iv) there exists in E a unique element, θ, called the **origin** or the (additive) **zero element**, such that for all x in E, $x \oplus \theta = x$; and
(v) to every x in E there corresponds a unique element, denoted by $-x$, such that $x \oplus (-x) = \theta$.

Scalar multiplication is an operation $\cdot$ which satisfies the following axioms:

(vi) For every pair consisting of a real (complex) number a and an element x in E, $a \cdot x$, called the **product** of a and x, is an element of E;
(vii) multiplication is distributive with respect to addition in E: $a \cdot (x \oplus y) = a \cdot x \oplus a \cdot y$;
(viii) multiplication is distributive with respect to the addition of real (complex) numbers: $(a + b) \cdot x = a \cdot x \oplus b \cdot x$;
(ix) multiplication is associative: $a \cdot (b \cdot x) = (ab) \cdot x$;
(x) $1 \cdot x = x$ for all x in E.

From the axioms it follows that the set E with the operation addition is an abelian group and that multiplication by a fixed scalar is an endomorphism of this group.

In the axioms, $+$ and juxtaposition denote respectively addition and multiplication of real (complex) numbers. Because of the relations between the two kinds of addition and the two kinds of multiplication, no confusion results from the practice, to be followed henceforth, of denoting both kinds of addition by $+$ and both kinds of multiplication by juxtaposition. Also, henceforth 0 denotes, ambiguously, either zero or the additive zero element θ of the abelian group formed by the elements of E and addition. Furthermore, it is customary to say simply "the linear space E" without reference to the operations. The elements of a linear space E are called **vectors**. The **scalar field** K of a real (complex) linear space is the field of real (complex) numbers, and its elements are frequently called **scalars.** The real (complex) field is itself a linear space under the convention that vector addition is ordinary addition, and that scalar multiplication is ordinary multiplication in the field. If it is said that a linear space E is the real (complex) field, it will always be understood that E is a linear space in this sense.

Two linear spaces E and F are identical if and only if $E = F$ and also the operations of addition and scalar multiplication are the same. In particular, the real linear space obtained from a complex linear space by restricting the domain of scalar multiplication to the real numbers is distinct from the latter space. It is called the **real restriction** of the complex linear space. It must be emphasized that the real restriction of a complex linear space has the same set of elements and the same operation of addition; moreover, scalar multiplication in the complex space and its real restriction coincide when both are defined. The only difference—but it is an important difference—is that the domain of the scalar multiplication of the real restriction is a proper subset of the domain of the original scalar multiplication. The real restriction of the complex field is the two-dimensional Euclidean space. (By definition, **real (complex) Euclidean** n-space is the space of all n-tuples of real (complex, respectively) numbers, with addition and scalar multiplication defined coordinatewise.) It may be observed that not every real linear space is the real restriction of a complex linear space (for example, one-dimensional real Euclidean space).

A subset A of a linear space E is **(finitely) linearly independent** if and only if a finite linear combination $\sum \{a_i x_i : i = 1, \cdots, n\}$, where $x_i \in A$ for each i and $x_i \neq x_j$ for $i \neq j$, is 0 only when each a_i is zero. This is equivalent to requiring that each member of E which can be written as a linear combination, with non-zero coefficients, of distinct members of A have a unique such representation (the difference of two distinct representations exhibits linear dependence of A). A subset B of E is a **Hamel base** for E if and only if each non-zero element of E is representable in a unique way as a finite linear combination of distinct members of B, with non-zero coefficients. A Hamel base is necessarily linearly independent, and the next theorem shows that any linearly independent set can be expanded to give a Hamel base.

1.1 THEOREM *Let E be a linear space. Then:*

(i) *Each linearly independent subset of E is contained in a maximal linearly independent subset.*

(ii) *Each maximal linearly independent subset is a Hamel base, and conversely.*

(iii) *Any two Hamel bases have the same cardinal number.*