LINEAR ALGEBRA

BY

WERNER H. GREUB

SECOND EDITION

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WITH 6 FIGURES

SPRINGER-VERLAG
BERLIN · GÖTTINGEN · HEIDELBERG
1963

Preface to the second edition

Besides the very obvious change from German to English, the second edition of this book contains many additions as well as a great many other changes. It might even be called a new book altogether were it not for the fact that the essential character of the book has remained the same; in other words, the entire presentation continues to be based on an axiomatic treatment of linear spaces.

In this second edition, the thorough-going restriction to linear spaces of finite dimension has been removed. Another complete change is the restriction to linear spaces with real or complex coefficients, thereby removing a number of relatively involved discussions which did not really contribute substantially to the subject. On p. 6 there is a list of those chapters in which the presentation can be transferred directly to spaces over an arbitrary coefficient field.

Chapter I deals with the general properties of a linear space. Those concepts which are only valid for finitely many dimensions are discussed in a special paragraph.

Chapter II now covers only linear transformations while the treatment of matrices has been delegated to a new chapter, chapter III. The discussion of dual spaces has been changed; dual spaces are now introduced abstractly and the connection with the space of linear functions is not established until later.

Chapters IV and V, dealing with determinants and orientation respectively, do not contain substantial changes. Brief reference should be made here to the new paragraph in chapter IV on the trace of an endomorphism — a concept which is used quite consistently throughout the book from that time on.

Special emphasize is given to tensors. The original chapter on Multilinear Algebra is now spread over four chapters: Multilinear Mappings (Ch. VI), Tensor Algebra (Ch. VII), Exterior Algebra (Ch. VIII) and Duality in Exterior Algebra (Ch. IX). The chapter on multilinear mappings consists now primarily of an introduction to the theory of the tensor-product. In chapter VII the notion of vector-valued tensors has been introduced and used to define the contraction. Furthermore, a treatment of the transformation of tensors under linear mappings has been added. In Chapter VIII the antisymmetry-operator is studied in greater detail and the concept of the skew-symmetric power is introduced. The dual product (Ch. IX) is generalized to mixed tensors. A special paragraph in this chapter covers the skew-symmetric powers of the unit tensor and shows their significance in the characteristic polynomial. The paragraph "Adjoint Tensors" provides a number of applications of the duality theory to certain tensors arising from an endomorphism of the underlying space.

There are no essential changes in Chapter X (Inner product spaces) except for the addition of a short new paragraph on normed linear spaces. In the next chapter, on linear mappings of inner product spaces, the orthogonal projections (§ 3) and the skew mappings (§ 4) are discussed in greater detail. Furthermore, a paragraph on differentiable families of automorphisms has been added here.

Chapter XII (Symmetric Bilinear Functions) contains a new paragraph dealing with Lorentz-transformations.

Whereas the discussion of quadrics in the first edition was limited to quadrics with centers, the second edition covers this topic in full.

The chapter on unitary spaces has been changed to include a more thorough-going presentation of unitary transformations of the complex plane and their relation to the algebra of quaternions.

The restriction to linear spaces with complex or real coefficients has of course greatly simplified the construction of irreducible subspaces in chapter XV. Another essential simplification of this construction was achieved by the simultaneous consideration of the dual mapping. A final paragraph with applications to Lorentz-transformation has been added to this concluding chapter.

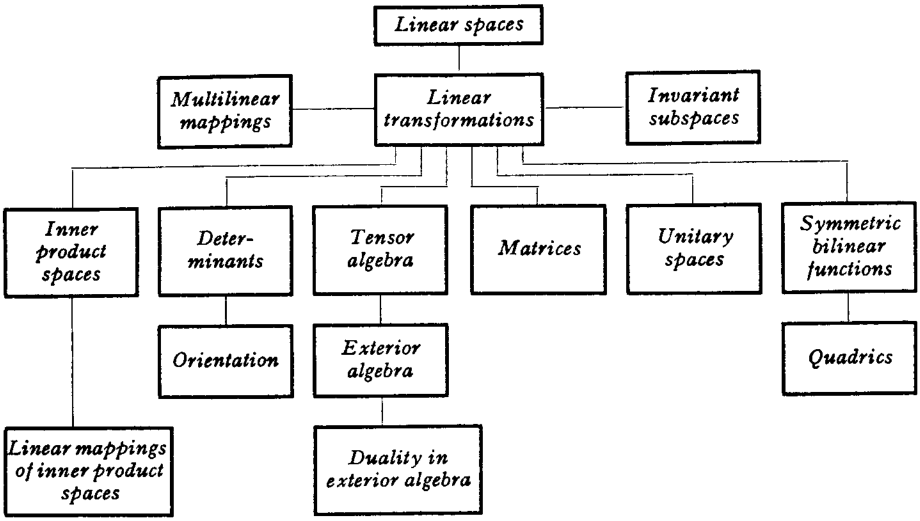
Many other minor changes have been incorporated — not least of which are the many additional problems now accompanying each paragraph.

Last, but certainly not least, I have to express my sincerest thanks to everyone who has helped me in the preparation of this second edition. First of all, I am particularly indebted to Cornelie J. Rheinboldt who assisted in the entire translating and editing work and to Dr. Werner C. Rheinboldt who cooperated in this task and who also made a number of valuable suggestions for improvements, especially in the chapters on linear transformations and matrices. My warm thanks also go to Dr. H. Bolder of the Royal Dutch/Shell Laboratory at Amsterdam for his criticism on the chapter on tensor-products and to Dr. H. H. Keller who read the entire manuscript and offered many important suggestions. Furthermore, I am grateful to Mr. Giorgio Pederzoli who helped to read the proofs of the entire work and who collected a number of new problems and to Mr. Khadja Nesamuddin Khan for his assistance in preparing the manuscript.

Finally I would like to express my thanks to the publishers for their patience and cooperation during the preparation of this edition.

Toronto, April 1963 Werner H. Greub

Interdependence of Chapters



Contents

Chapter I

Linear spaces

§ 1. The axioms of a linear space

1.1. Additive groups. A set E: $(x, y \dots)$ is called an *additive group* if to every pair x and y there is assigned a third element of E, called the *sum* of x and y and written as $x + y$, such that the following axioms hold:

I.1. $x + y = y + x$ (commutative law)

I.2. $(x + y) + z = x + (y + z)$ (associative law)

I.3. There exists a zero-element 0 such that $x + 0 = x$ for every $x \in E$.

I.4. To every element x there exists an *inverse element* $-x$ such that $x + (-x) = 0$.

The zero-element is uniquely determined. In fact, assume there are two such elements 0 and 0′. Then for every $x \in E$

$$x + 0 = x \quad \text{and} \quad x + 0' = x .$$

Substituting $x = 0'$ in the first and $x = 0$ in the second equation we obtain

$$0' + 0 = 0' \quad \text{and} \quad 0 + 0' = 0$$

and hence by the commutative law $0 = 0'$.

For every $x \in E$, there is only one inverse element $-x$. We prove more generally that to any two elements a and b there is exactly one element x such that

$$x + a = b . \tag{1.1}$$

To show first the uniqueness, let x_1 and x_2 be two solutions of (1.1). Then

$$x_1 + a = b \quad \text{and} \quad x_2 + a = b$$

and consequently

$$x_1 + a = x_2 + a .$$

Now let $-a$ be a negative element of a. Adding $-a$ to the above equation we obtain by the associative law

$$x_1 + (a + (-a)) = x_2 + (a + (-a))$$

and hence $x_1 = x_2$. This result applied to the vector $b = 0$ yields the uniqueness of $-a$.

To show that (1.1) always has a solution, consider the element

$$x = b + (-a)\,. \tag{1.2}$$

Then

$$x + a = b + (-a) + a = b + 0 = b\,.$$

The element x defined by (1.2) is called the *difference* of b and a and is denoted by $b - a$.

1.2. Real linear spaces. A *real linear space* or *real vector space* is an additive group with the following additional structure: There is defined a multiplication between the real numbers $\lambda, \mu \ldots$ and the elements of E; in other words, to every pair (λ, x) an element λx of E is assigned, subject to the following axioms:

II.1. $(\lambda\,\mu)\,x = \lambda(\mu\,x)$ (associative law)

II.2. $(\lambda + \mu)\,x = \lambda x + \mu x$, $\lambda(x + y) = \lambda x + \lambda y$ (distributive laws)

II.3. $1 \cdot x = x$.

The elements of a linear space are called *vectors* and the coefficients *scalars*. From the first distributive law II.2 we obtain by inserting $\mu = 0$

$$\lambda x = \lambda x + 0 \cdot x$$

and adding the vector $-(\lambda x)$

$$0 \cdot x = 0\,^{*)}.$$

Similarly, the second law II.2. yields for $y = 0$

$$\lambda \cdot 0 = 0\,.$$

These two equations state that $\lambda x = 0$ if $\lambda = 0$ or $x = 0$. Conversely, the equation $\lambda x = 0$ implies that $\lambda = 0$ or $x = 0$. In fact, assume that $\lambda \neq 0$. Then it follows from the axioms II.3 and II.1. that

$$x = 1 \cdot x = \left(\frac{1}{\lambda} \cdot \lambda\right) x = \frac{1}{\lambda}\,(\lambda x) = \frac{1}{\lambda} \cdot 0 = 0\,.$$

Altogether we have shown that $\lambda x = 0$ if and only if $\lambda = 0$ or $x = 0$.

Substituting $\mu = -\lambda$ in the first distributive law we obtain

$$\lambda x + (-\lambda x) = 0\,,$$

whence

$$(-\lambda)\,x = -\lambda x\,.$$

Similarly, the second distributive law yields

$$\lambda(-x) = -\lambda x\,.$$

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*) It should be observed that the symbol 0 on the left-hand side denotes the *scalar* zero and on the right-hand side the *vector* zero.

Finally we observe that the two distributive laws hold for any finite number of terms,

$$\left(\sum_\nu \lambda^\nu\right) x = \sum_\nu \lambda^\nu x$$

$$\lambda \sum_\nu x_\nu = \sum_\nu \lambda x_\nu$$

as can be shown by induction.

1.3. Examples: 1. Consider the set of all ordered n-tuples of real numbers

$$x = (\xi^1 \dots \xi^n)$$

where n is a fixed integer. Addition of two n-tuples

$$x = (\xi^1 \dots \xi^n) \quad \text{and} \quad y = (\eta^1 \dots \eta^n)$$

is defined by

$$x + y = (\xi^1 + \eta^1 \dots \xi^n + \eta^n)$$

and multiplication by a real number by

$$\lambda x = (\lambda \xi^1, \dots \lambda \xi^n) .$$

The linear space thus obtained is called the real *n-dimensional number-space* and is denoted by R^n. Its zero-vector is the n-tuple

$$0 = (0 \dots 0)$$

and the inverse of a vector x is given by the n-tuple

$$-x = (-\xi^1 \dots -\xi^n) .$$

2. Denote by C the set of all real valued continuous functions f in the interval $0 \leqq t \leqq 1$. Defining addition and multiplication by a real number as

$$(f + g)(t) = f(t) + g(t)$$

and

$$(\lambda f)(t) = \lambda f(t)$$

we obtain a linear space. The zero-vector of this linear space is the identically vanishing function.

Instead of all continuous functions we could also consider the set of all differentiable functions or the set of all continuously differentiable functions.

3. Let S be an arbitrary set. Consider all real valued functions in S which assume the value zero except for finitely many points of S. If addition and multiplication is defined as in example 2 this set becomes a linear space $C(S)$. For every element $a \in S$ denote by f_a the function defined by

$$f_a(x) = \begin{cases} 1 \text{ if } x = a \\ 0 \text{ if } x \neq a . \end{cases}$$

Then every function $f \in C(S)$ can be written as a finite linear combination

$$f = \sum_{a \in A} f(a) \cdot f_a$$

where A is the set of all points x for which $f(x) \neq 0$. Identifying every point $a \in S$ with the corresponding function f_a we can write the above equation as

$$f = \sum_{a \in A} f(a) \cdot a .$$

In this notation a function $f \in C(S)$ appears as a "formal linear combination" of the elements of S.

The space $C(S)$ is called the *linear space generated by* S.

1.4. Linear dependence. A system of p ($p \geqq 1$) vectors $(x_1 \dots x_p)$ of a linear space E is called *linearly dependent* if there exist coefficients $(\lambda^1 \dots \lambda^p)$, not all zero, such that

$$\sum_\nu \lambda^\nu x_\nu = 0 .$$

Otherwise the vectors $(x_1 \dots x_p)$ are called *linearly independent*. One single vector x is obviously linearly dependent if and only if $x = 0$.

If the system $(x_1 \dots x_p)$ is linearly dependent, so is every other system $(x_1 \dots x_p \dots x_q)$ containing the vectors $x_1 \dots x_p$. In fact, assume that

$$\lambda^1 x_1 + \cdots + \lambda^p x_p = 0$$

with at least one $\lambda^\nu \neq 0$. Then the relation

$$\lambda^1 x_1 + \cdots + \lambda^p x_p + 0 \cdot x_{p+1} + \cdots 0 \cdot x_q = 0$$

shows that the vectors $(x_1 \dots x_q)$ are again linearly dependent. In particular, every system containing the zero-vector is linearly dependent.

From this result it follows that a system of linearly independent vectors remains linearly independent if some vectors are omitted.

1.5. Cartesian Product. Consider two linear spaces E and F. Form the product set $E \times F$ defined as the set of all pairs (x, y) with $x \in E$ and $y \in F$. In $E \times F$ introduce addition and multiplication by real numbers as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$\lambda (x, y) = (\lambda x, \lambda y) .$$

It is easy to see that these two operations satisfy the axioms of a linear space. The space $E \times F$ thus obtained is called the *Cartesian product* of E and F. In the same way the Cartesian product of any finite number of linear spaces can be defined.

1.6. Complex linear spaces. Instead of using real numbers as coefficients in a linear space, one can also take complex numbers. In this way one obtains a *complex linear space*. More precisely, let E be an additive group.

Assume that to every complex number λ and every vector $x \in E$ a vector $\lambda x \in E$ is assigned such that the three axioms II of sec. 1.2 are satisfied. Then E is called a *complex linear space.*

As an example, consider the set of all ordered n-tuples of complex numbers

$$z = (\zeta^1 \ldots \zeta^n)$$

with operations defined as in sec. 1.3 for the real number-space. The complex linear space C^n thus obtained is called the *n-dimensional complex number-space.*

1.7. Linear spaces over an arbitrary coefficient-field. The only properties of the real or the complex numbers used in the axioms of a linear space are those based upon the additive and multiplicative structure of these numbers. This fact suggests the generalization of the concept of a linear space by using as scalars the elements of an arbitrary commutative coefficient-field.

A *commutative field* Λ is a set of elements $\alpha, \beta \ldots$ with two operations, addition and multiplication, subject to the following conditions:

I. Laws of addition:

1. $\alpha + \beta = \beta + \alpha$ (commutative law).
2. $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ (associative law).
3. There exists an element 0 such that $\alpha + 0 = \alpha$ for every element $\alpha \in \Lambda$.
4. To every element α there exists an element $-\alpha$ such that $\alpha + (-\alpha) = 0$.

The above axioms assert that Λ is an abelian group.

II. Laws of multiplication:

1. $\alpha\beta = \beta\alpha$ (commutative law).
2. $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ (associative law).
3. There exists an element $\varepsilon \in \Lambda$ such that $\varepsilon\alpha = \alpha$ for every $\alpha \in \Lambda$.
4. To every element $\alpha \neq 0$ there exists an element α^{-1} such that $\alpha\alpha^{-1} = \varepsilon$.

III. The distributive law:

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma.$$

1.8. Let Λ be a given commutative field. A *linear space over the coefficient-field* Λ is an additive group in which a multiplication

$$(\lambda, x) \rightarrow \lambda x, \quad \lambda \in \Lambda, \quad x \in E$$

is defined such that the axioms II of sec. 1.2 are satisfied*). All properties derived from these axioms remain true for a linear space over an arbitrary commutative coefficient-field.

*) The number 1 in axiom II.3 has to be replaced by ε.

For the sake of simplicity we shall concern ourselves in the following chapters only with real and complex linear spaces. In other words, whenever we speak about linear spaces without further specification, a real or a complex space is understood. However, all the developments of the chapters I, II, III, VI*) and VII apply word for word to a linear space over an arbitrary commutative coefficient-field. The results of the chapters IV, VIII and IX can be carried over to linear spaces over a commutative coefficient-field, whose characteristic**) is different from 2.

Problems: 1. Show:

a) The set of all real numbers of the form $a + b\sqrt{5}$ with a and b integers forms an additive group,

b) the set of all real numbers of the form $\alpha + \beta\sqrt{3}$ with α and β rational forms a field,

c) the set of all complex numbers of the forms $\gamma + i\delta$ where γ and δ are real and $i = \sqrt{-1}$ forms a field.

2. Show that axiom II.3 can be replaced by the following one: The equation $\lambda x = 0$ holds if and only if $\lambda = 0$ or $x = 0$.

3. Given a system of linearly independent vectors $(x_1, \ldots, x_p)$, prove that the system $x_1 \ldots x_i + \lambda x_j, \ldots, x_p (i \neq j)$ with arbitrary λ is again linearly independent.

4. Show that the set of all solutions of the homogeneous linear differential equation

$$\frac{d^2 y}{dt^2} + p\frac{dy}{dt} + qy = 0,$$

where p and q are functions of t, is a vector space.

5. Which of the following sets of functions are linearly dependent?

a) $f_1 = 3t$; $f_2 = t + 5$; $f_3 = 2t^2$; $f_4 = (t+1)^2$

b) $f_1 = (t+1)^2$; $f_2 = t^2 - 1$; $f_3 = 2t^2 + 2t - 3$

c) $f_1 = 1$; $f_2 = e^{it}$; $f_3 = e^{-it}$

d) $f_1 = t^2$; $f_2 = t$; $f_3 = 1$

e) $f_1 = 1 - t$; $f_2 = t(1-t)$; $f_3 = 1 - t^2$.

6. Let E be a real linear space. Consider the set $E \times E$ of ordered pairs (x, y) with $x \in E$ and $y \in E$. Show that the set $E \times E$ becomes a complex linear space C by the operations:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and

$$(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \alpha y + \beta x) \quad (\alpha, \beta \text{ real numbers}).$$

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*) Except for the remarks about skew-symmetric mappings in sec 6.4.

**) Concerning the definition of the characteristic cf. VAN DER WAERDEN, Algebra.

7. Are the vectors $x_1 = (1, 0, 1)$; $x_2 = (i, 1, 0)$; $x_3 = (i, 2, 1+i)$ linearly independent in C^3? Express $x_4 = (1, 2, 3)$ as a linear combination of x_1, x_2, x_3. Do the same for $x_5 = (i, i, i)$.

§ 2. Linear subspaces

1.9. Definition. A nonempty subset E_1 of a linear space E is called a *linear subspace* if the following conditions hold:

1. If $x \in E_1$ and $y \in E_1$, then $x + y \in E_1$.
2. If $x \in E_1$, then $\lambda x \in E_1$ for every coefficient λ.

The two above conditions are equivalent to the condition that E_1, with any two vectors x and y, contains all linear combinations $\lambda x + \mu y$.

By substituting $\lambda = 0$ in the second condition it follows that every subspace contains the zero-vector. A subspace E_1 is called *improper*, if E_1 consists of the entire space E or if E_1 reduces to the zero-vector; otherwise E_1 is called a *proper* subspace.

Every nonempty set S in E determines a subspace called the *linear closure* of S. It consists of all possible finite linear combinations

$$x = \sum_\nu \xi^\nu x_\nu \qquad x_\nu \in S$$

with arbitrary coefficients ξ^ν.

1.10. Intersection and sum. Let E_1 and E_2 be two subspaces of E. Then the set of all vectors contained in E_1 and in E_2 is again a linear subspace. This subspace is called the *intersection* of E_1 and E_2 and is denoted by $E_1 \cap E_2$.

The *sum* of E_1 and E_2, denoted as $E_1 + E_2$, is the set of all vectors

$$x = x_1 + x_2, \quad x_1 \in E_1, \quad x_2 \in E_2{}^{*)}.$$

Obviously, $E_1 + E_2$ is a subspace of E, containing E_1 and E_2 as subspaces.

A vector x of the sum $E_1 + E_2$ can generally be decomposed as $x = x_1 + x_2$ $(x_1 \in E_1, x_2 \in E_2)$ in different ways. Given two decompositions

$$x = x_1 + x_2, \quad x_1 \in E_1, \quad x_2 \in E_2$$

and

$$x = x_1' + x_2', \quad x_1' \in E_1, \quad x_2' \in E_2$$

it follows that

$$x_1 - x_1' = x_2' - x_2.$$

Hence, the vector

$$z = x_1 - x_1'$$

is contained in the intersection $E_1 \cap E_2$. Conversely, let $x = x_1 + x_2$ be a

*) The sum $E_1 + E_2$ has to be distinguished from the set-theoretic union $E_1 \cup E_2$ which in general is not a linear space. $E_1 + E_2$ is obviously the linear closure of $E_1 \cup E_2$.

decomposition of x and z be an arbitrary vector of $E_1 \cap E_2$. Then the vectors

$$x_1' = x_1 - z \quad \text{and} \quad x_2' = x_2 + z$$

form again a decomposition of x. It follows from this remark that the vectors x_1 and x_2 are uniquely determined by x if and only if the intersection $E_1 \cap E_2$ reduces to the zero-vector. In this case the space $E_1 + E_2$ is called the *direct sum* of E_1 and E_2 and is denoted by $E_1 \oplus E_2$.

In the same way as it is for two subspaces, the intersection and sum of finitely many subspaces E_i $(i = 1 \ldots p)$ is defined. If any two spaces E_i and E_j $(i \neq j)$ have only the zero-vector in common, the space $E_1 + \ldots + E_p$ is called the *direct sum* of the spaces E_i $(i = 1 \ldots p)$.

1.11. Factor-space. Let E_1 be a subspace of E. Then an equivalence relation among the vectors of E can be defined in the following way: Two vectors x and x' are to be equivalent, $x \sim x'$, if $x' - x \in E_1$. This relation has indeed the three properties of an equivalence:

1. *Reflexivity*: $x \sim x$ for every $x \in E$, since $x - x = 0 \in E_1$.

2. *Commutativity*: $x \sim x'$ implies that $x' \sim x$: If $x' - x \in E_1$, then $x - x' = -(x' - x) \in E_1$.

3. *Transitivity*: $x \sim x'$ and $x' \sim x''$ implies that $x \sim x''$: If $x' - x \in E_1$ and $x'' - x' \in E_1$, then $x'' - x = (x'' - x') + (x' - x) \in E_1$.

An equivalence relation induces a decomposition of the whole space into classes of equivalent vectors. Two vectors x and x' of E are in the same class if and only if they are equivalent. Any two classes C_1 and C_2 are either disjoint or they coincide. In fact, assume that $x \in C_1 \cap C_2$. Then

$$x \sim x_1 \quad \text{and} \quad x \sim x_2$$

for every vector $x_1 \in C_1$ and every vector $x_2 \in C_2$. This implies in view of the transitivity that $x_1 \sim x_2$, whence $C_1 = C_2$.

Thus, every vector $x \in E$ is contained in exactly one class. This class will be denoted by $\bar{x}$. The class $\bar{0}$ containing the zero-vector coincides with the subspace E_1. It should be observed that this is the only class which is itself a linear subspace of E since the other classes do not contain the zero-vector.

To get a geometric picture of the above decomposition let E be a linear space of three dimensions and E_1 be a plane through 0. Then the corresponding classes are the planes parallel to E_1.

1.12. The linear structure of the factor-space. Consider the set of all equivalence classes with respect to E_1. This set can be made into a linear space by defining the linear operations as follows: Let $\bar{x}$ and $\bar{y}$ be two classes. Choose two vectors $x \in \bar{x}$ and $y \in \bar{y}$. Then the vector $x + y$ is contained in a certain class $\overline{x + y}$. This class does not depend on the choice of x and y but only on the classes $\bar{x}$ and $\bar{y}$. In fact, taking two other representatives x' and y', we have the relations

$$x' - x \in E_1 \quad \text{and} \quad y' - y \in E_1$$

and hence,

$$(x' + y') - (x + y) = (x' - x) + (y' - y) \in E_1 .$$

This implies that

$$x' + y' \sim x + y$$

and consequently, that

$$\overline{x' + y'} = \overline{x + y} .$$

The class $\overline{x + y}$ therefore is uniquely determined by the classes $\bar{x}$ and $\bar{y}$ and so it is proper to call it the sum of $\bar{x}$ and $\bar{y}$:

$$\bar{x} + \bar{y} = \overline{x + y} .$$

Similarly, the product $\lambda\bar{x}$ is defined as the equivalence class of the vector λx where x is any representative of $\bar{x}$,

$$\lambda\bar{x} = \overline{\lambda x} .$$

As in the case of the addition, it follows that this class depends only on the class $\bar{x}$.

It is easily verified that the two operations so defined satisfy the axioms listed in sec. 1.1 and 1.2. Thus, the set of all classes becomes a linear space, called the *factor-space* of E with respect to E_1 and denoted by E/E_1. It is also usual to call E/E_1 the *quotient-space* of E with respect to E_1. The zero-vector of the factor-space is the class $\bar{0}$.

If the subspace E_1 coincides with E, all vectors are equivalent and hence there is only the class $\bar{0}$. In this case the factor-space E/E_1 reduces to the zero-vector. If, in the opposite case, E_1 consists only of the zero-vector , two vectors of E are equivalent if and only if they are equal and so each class consists of exactly one vector. In this case the factor-space coincides with E.

Problems: 1. Let (ξ^1, ξ^2, ξ^3) be an arbitrary vector in R^3. Which of the following subsets are subspaces?

a) All vectors with $\xi^1 = \xi^2 = \xi^3$,
b) all vectors with $\xi^3 = 0$,
c) all vectors with $\xi^1 = \xi^2 - \xi^3$,
d) all vectors with $\xi^2 = 1$.

2. Let S be an arbitrary subset of E and $\bar{S}$ its linear closure. Show that $\bar{S}$ is the intersection of all linear subspaces of E containing S.

3. Let $E = E_1 \oplus E_2$ and $F = F_1 \oplus F_2$ be two direct decompositions of the spaces E and F. Show that the Cartesian product $E \times F$ can be directly decomposed as follows:

$$E \times F = E_1 \times F_1 \oplus E_2 \times F_2 .$$

4. Assume a direct decomposition $E = E_1 \oplus E_2$. Show that in each class of E with respect to E_1 there is exactly one vector of E_2.

5. Let E be a plane and E_1 a straight line through the origin. What is the geometrical meaning of the equivalence classes respect to E_1? Give a geometrical interpretation of the fact that $x \sim x'$ and $y \sim y'$ implies $x + y \sim x' + y'$.

§ 3. Linear spaces of finite dimension

1.13. Dimension. In general there are infinitely many linearly independent vectors in a linear space. For instance, in the space of all continuous functions $f(t)$ $(0 \leqq t \leqq 1)$ all the powers $t, t^2, t^3 \ldots$ are linearly independent. We shall be mainly concerned with linear spaces having only finitely many linearly independent vectors. The maximal number of linearly independent vectors of such a space E is called the *dimension* of E and will be denoted by $\dim E$.

A 1-dimensional linear space is called *a straight line* and a 2-dimensional linear space is called a *plane*.

Let E be an n-dimensional linear space and E_1 be a subspace of E. Since every system of linearly independent vectors of E_1 is also linearly independent in E, the dimension of E_1 can be at most equal to the dimension of E,

$$\dim E_1 \leqq \dim E .$$

It will be shown in the next section that the equality holds only if $E_1 = E$.

1.14. Basis. Let E be a linear space of dimension n. A system of n linearly independent vectors is called a *basis* of E. Then every vector x can be uniquely represented as a linear combination

$$x = \sum_\nu \xi^\nu x_\nu .$$

In fact, consider the $n + 1$ vectors $x_1 \ldots x_n, x$. These must be linearly dependent because there can be at most n linearly independent vectors in E. Therefore a relation

$$\sum_\nu \lambda^\nu x_\nu + \lambda x = 0 \tag{1.3}$$

holds with at least one coefficient different from zero. In particular, $\lambda \neq 0$, since otherwise (1.3) reduces to

$$\sum_\nu \lambda^\nu x_\nu = 0$$

which implies that $\lambda^\nu = 0 \, (\nu = 1 \ldots n)$. Thus the equation (1.3) can be solved with respect to x, yielding

$$x = -\frac{1}{\lambda} \sum_\nu \lambda^\nu x_\nu = \sum_\nu \xi^\nu x_\nu . \tag{1.4}$$