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Algebraic Geometry

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Preface

This book provides an introduction to abstract algebraic geometry using the methods of schemes and cohomology. The main objects of study are algebraic varieties in an affine or projective space over an algebraically closed field; these are introduced in Chapter I, to establish a number of basic concepts and examples. Then the methods of schemes and cohomology are developed in Chapters II and III, with emphasis on applications rather than excessive generality. The last two chapters of the book (IV and V) use these methods to study topics in the classical theory of algebraic curves and surfaces.

The prerequisites for this approach to algebraic geometry are results from commutative algebra, which are stated as needed, and some elementary topology. No complex analysis or differential geometry is necessary. There are more than four hundred exercises throughout the book, offering specific examples as well as more specialized topics not treated in the main text. Three appendices present brief accounts of some areas of current research.

This book can be used as a textbook for an introductory course in algebraic geometry, following a basic graduate course in algebra. I recently taught this material in a five-quarter sequence at Berkeley, with roughly one chapter per quarter. Or one can use Chapter I alone for a short course. A third possibility worth considering is to study Chapter I, and then proceed directly to Chapter IV, picking up only a few definitions from Chapters II and III, and *assuming* the statement of the Riemann–Roch theorem for curves. This leads to interesting material quickly, and may provide better motivation for tackling Chapters II and III later.

The material covered in this book should provide adequate preparation for reading more advanced works such as Grothendieck [EGA], [SGA], Hartshorne [5], Mumford [2], [5], or Shafarevich [1].

Acknowledgements

In writing this book, I have attempted to present what is essential for a basic course in algebraic geometry. I wanted to make accessible to the nonspecialist an area of mathematics whose results up to now have been widely scattered, and linked only by unpublished "folklore." While I have reorganized the material and rewritten proofs, the book is mostly a synthesis of what I have learned from my teachers, my colleagues, and my students. They have helped in ways too numerous to recount. I owe especial thanks to Oscar Zariski, J.-P. Serre, David Mumford, and Arthur Ogus for their support and encouragement.

Aside from the "classical" material, whose origins need a historian to trace, my greatest intellectual debt is to A. Grothendieck, whose treatise [EGA] is the authoritative reference for schemes and cohomology. His results appear without specific attribution throughout Chapters II and III. Otherwise I have tried to acknowledge sources whenever I was aware of them.

In the course of writing this book, I have circulated preliminary versions of the manuscript to many people, and have received valuable comments from them. To all of these people my thanks, and in particular to J.-P. Serre, H. Matsumura, and Joe Lipman for their careful reading and detailed suggestions.

I have taught courses at Harvard and Berkeley based on this material, and I thank my students for their attention and their stimulating questions.

I thank Richard Bassein, who combined his talents as mathematician and artist to produce the illustrations for this book.

A few words cannot adequately express the thanks I owe to my wife, Edie Churchill Hartshorne. While I was engrossed in writing, she created a warm home for me and our sons Jonathan and Benjamin, and through her constant support and friendship provided an enriched human context for my life.

For financial support during the preparation of this book, I thank the Research Institute for Mathematical Sciences of Kyoto University, the National Science Foundation, and the University of California at Berkeley.

August 29, 1977
Berkeley, California

ROBIN HARTSHORNE

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Introduction

The author of an introductory book on algebraic geometry has the difficult task of providing geometrical insight and examples, while at the same time developing the modern technical language of the subject. For in algebraic geometry, a great gap appears to separate the intuitive ideas which form the point of departure from the technical methods used in current research.

The first question is that of language. Algebraic geometry has developed in waves, each with its own language and point of view. The late nineteenth century saw the function-theoretic approach of Riemann, the more geometric approach of Brill and Noether, and the purely algebraic approach of Kronecker, Dedekind, and Weber. The Italian school followed with Castelnuovo, Enriques, and Severi, culminating in the classification of algebraic surfaces. Then came the twentieth-century "American" school of Chow, Weil, and Zariski, which gave firm algebraic foundations to the Italian intuition. Most recently, Serre and Grothendieck initiated the French school, which has rewritten the foundations of algebraic geometry in terms of schemes and cohomology, and which has an impressive record of solving old problems with new techniques. Each of these schools has introduced new concepts and methods. In writing an introductory book, is it better to use the older language which is closer to the geometric intuition, or to start at once with the technical language of current research?

The second question is a conceptual one. Modern mathematics tends to obliterate history: each new school rewrites the foundations of its subject in its own language, which makes for fine logic but poor pedagogy. Of what use is it to know the definition of a scheme if one does not realize that a ring of integers in an algebraic number field, an algebraic curve, and a compact Riemann surface are all examples of a "regular scheme of dimension one"? How then can the author of an introductory book indicate the inputs to algebraic geometry coming from number theory, commutative algebra, and complex analysis, and also introduce the reader to the main objects of study, which are algebraic varieties in affine or projective space, while at the same time developing the modern language of schemes and cohomology? What choice of topics will convey the meaning of algebraic geometry, and still serve as a firm foundation for further study and research?

My own bias is somewhat on the side of classical geometry. I believe that the most important problems in algebraic geometry are those arising from old-fashioned varieties in affine or projective spaces. They provide the geometric intuition which motivates all further developments. In this book, I begin with a chapter on varieties, to establish many examples and basic ideas in their simplest form, uncluttered with technical details. Only after that do I develop systematically the language of schemes, coherent sheaves, and cohomology, in Chapters II and III. These chapters form the technical heart of the book. In them I attempt to set forth the most important results, but without striving for the utmost generality. Thus, for example, the cohomology theory is developed only for quasi-coherent sheaves on noetherian schemes, since this is simpler and sufficient for most applications; the theorem of "coherence of direct image sheaves" is proved only for projective morphisms, and not for arbitrary proper morphisms. For the same reasons I do not include the more abstract notions of representable functors, algebraic spaces, étale cohomology, sites, and topoi.

The fourth and fifth chapters treat classical material, namely nonsingular projective curves and surfaces, but they use techniques of schemes and cohomology. I hope these applications will justify the effort needed to absorb all the technical apparatus in the two previous chapters.

As the basic language and logical foundation of algebraic geometry, I have chosen to use commutative algebra. It has the advantage of being precise. Also, by working over a base field of arbitrary characteristic, which is necessary in any case for applications to number theory, one gains new insight into the classical case of base field **C**. Some years ago, when Zariski began to prepare a volume on algebraic geometry, he had to develop the necessary algebra as he went. The task grew to such proportions that he produced a book on commutative algebra only. Now we are fortunate in having a number of excellent books on commutative algebra: Atiyah–Macdonald [1], Bourbaki [1], Matsumura [2], Nagata [7], and Zariski–Samuel [1]. My policy is to quote purely algebraic results as needed, with references to the literature for proof. A list of the results used appears at the end of the book.

Originally I had planned a whole series of appendices—short expository accounts of some current research topics, to form a bridge between the main text of this book and the research literature. Because of limited time and space only three survive. I can only express my regret at not including the others, and refer the reader instead to the Arcata volume (Hartshorne, ed. [1]) for a series of articles by experts in their fields, intended for the nonspecialist. Also, for the historical development of algebraic geometry let me refer to Dieudonné [1]. Since there was not space to explore the relation of algebraic geometry to neighboring fields as much as I would have liked, let me refer to the survey article of Cassels [1] for connections with number theory, and to Shafarevich [2, Part III] for connections with complex manifolds and topology.

Because I believe strongly in active learning, there are a great many exercises in this book. Some contain important results not treated in the main text. Others contain specific examples to illustrate general phenomena. I believe that the study of particular examples is inseparable from the development of general theories. The serious student should attempt as many as possible of these exercises, but should not expect to solve them immediately. Many will require a real creative effort to understand. An asterisk denotes a more difficult exercise. Two asterisks denote an unsolved problem.

See (I, §8) for a further introduction to algebraic geometry and this book.

Terminology

For the most part, the terminology of this book agrees with generally accepted usage, but there are a few exceptions worth noting. A *variety* is always irreducible and is always over an algebraically closed field. In Chapter I all varieties are quasi-projective. In (Ch. II, §4) the definition is expanded to include *abstract varieties,* which are integral separated schemes of finite type over an algebraically closed field. The words *curve, surface,* and *3-fold* are used to mean varieties of dimension 1, 2, and 3 respectively. But in Chapter IV, the word *curve* is used only for a nonsingular projective curve; whereas in Chapter V a *curve* is any effective divisor on a nonsingular projective surface. A *surface* in Chapter V is always a nonsingular projective surface.

A *scheme* is what used to be called a prescheme in the first edition of [EGA], but is called scheme in the new edition of [EGA, Ch. I].

The definitions of a *projective morphism* and a *very ample invertible sheaf* in this book are not equivalent to those in [EGA]—see (II, §4, 5). They are technically simpler, but have the disadvantage of not being local on the base.

The word *nonsingular* applies only to varieties; for more general schemes, the words *regular* and *smooth* are used.

Results from algebra

I assume the reader is familiar with basic results about rings, ideals, modules, noetherian rings, and integral dependence, and is willing to accept or look up other results, belonging properly to commutative algebra or homological algebra, which will be stated as needed, with references to the literature. These results will be marked with an A: e.g., Theorem 3.9A, to distinguish them from results proved in the text.

The basic conventions are these: All rings are commutative with identity element 1. All homomorphisms of rings take 1 to 1. In an integral domain or a field, $0 \neq 1$. A *prime ideal* (respectively, *maximal ideal*) is an ideal $\mathfrak{p}$ in a ring A such that the quotient ring $A/\mathfrak{p}$ is an integral domain (respectively, a field). Thus the ring itself is not considered to be a prime ideal or a maximal ideal.

A *multiplicative system* in a ring A is a subset S, containing 1, and closed under multiplication. The *localization* $S^{-1}A$ is defined to be the ring formed by equivalence classes of fractions a/s, $a \in A$, $s \in S$, where a/s and a'/s' are said to be *equivalent* if there is an $s'' \in S$ such that $s''(s'a - sa') = 0$ (see e.g. Atiyah–Macdonald [1, Ch. 3]). Two special cases which are used constantly are the following. If $\mathfrak{p}$ is a prime ideal in A, then $S = A - \mathfrak{p}$ is a multiplicative system, and the corresponding localization is denoted by $A_{\mathfrak{p}}$. If f is an element of A, then $S = \{1\} \cup \{f^n \mid n \geq 1\}$ is a multiplicative system, and the corresponding localization is denoted by A_f. (Note for example that if f is nilpotent, then A_f is the zero ring.)

References

Bibliographical references are given by author, with a number in square brackets to indicate which work, e.g. Serre, [3, p. 75]. Cross references to theorems, propositions, lemmas within the same chapter are given by number in parentheses, e.g. (3.5). Reference to an exercise is given by (Ex. 3.5). References to results in another chapter are preceded by the chapter number, e.g. (II, 3.5), or (II, Ex. 3.5).

Contents

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CHAPTER I

Varieties

Our purpose in this chapter is to give an introduction to algebraic geometry with as little machinery as possible. We work over a fixed algebraically closed field k. We define the main objects of study, which are algebraic varieties in affine or projective space. We introduce some of the most important concepts, such as dimension, regular functions, rational maps, nonsingular varieties, and the degree of a projective variety. And most important, we give lots of specific examples, in the form of exercises at the end of each section. The examples have been selected to illustrate many interesting and important phenomena, beyond those mentioned in the text. The person who studies these examples carefully will not only have a good understanding of the basic concepts of algebraic geometry, but he will also have the background to appreciate some of the more abstract developments of modern algebraic geometry, and he will have a resource against which to check his intuition. We will continually refer back to this library of examples in the rest of the book.

The last section of this chapter is a kind of second introduction to the book. It contains a discussion of the "classification problem," which has motivated much of the development of algebraic geometry. It also contains a discussion of the degree of generality in which one should develop the foundations of algebraic geometry, and as such provides motivation for the theory of schemes.

1 Affine Varieties

Let k be a fixed algebraically closed field. We define *affine n-space* over k, denoted $\mathbf{A}_k^n$ or simply $\mathbf{A}^n$, to be the set of all n-tuples of elements of k. An element $P \in \mathbf{A}^n$ will be called a *point*, and if $P = (a_1, \ldots ,a_n)$ with $a_i \in k$, then the a_i will be called the *coordinates* of P.

Let $A = k[x_1, \ldots ,x_n]$ be the polynomial ring in n variables over k. We will interpret the elements of A as functions from the affine n-space to k, by defining $f(P) = f(a_1, \ldots ,a_n)$, where $f \in A$ and $P \in \mathbf{A}^n$. Thus if $f \in A$ is a polynomial, we can talk about the set of *zeros* of f, namely $Z(f) = \{P \in \mathbf{A}^n | f(P) = 0\}$. More generally, if T is any subset of A, we define the *zero set* of T to be the common zeros of all the elements of T, namely

$$Z(T) = \{P \in \mathbf{A}^n | f(P) = 0 \text{ for all } f \in T\}.$$

Clearly if $\mathfrak{a}$ is the ideal of A generated by T, then $Z(T) = Z(\mathfrak{a})$. Furthermore, since A is a noetherian ring, any ideal $\mathfrak{a}$ has a finite set of generators $f_1, \ldots ,f_r$. Thus $Z(T)$ can be expressed as the common zeros of the finite set of polynomials $f_1, \ldots ,f_r$.

Definition. A subset Y of $\mathbf{A}^n$ is an *algebraic set* if there exists a subset $T \subseteq A$ such that $Y = Z(T)$.

Proposition 1.1. *The union of two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets.*

PROOF. If $Y_1 = Z(T_1)$ and $Y_2 = Z(T_2)$, then $Y_1 \cup Y_2 = Z(T_1T_2)$, where T_1T_2 denotes the set of all products of an element of T_1 by an element of T_2. Indeed, if $P \in Y_1 \cup Y_2$, then either $P \in Y_1$ or $P \in Y_2$, so P is a zero of every polynomial in T_1T_2. Conversely, if $P \in Z(T_1T_2)$, and $P \notin Y_1$ say, then there is an $f \in T_1$ such that $f(P) \neq 0$. Now for any $g \in T_2$, $(fg)(P) = 0$ implies that $g(P) = 0$, so that $P \in Y_2$.

If $Y_\alpha = Z(T_\alpha)$ is any family of algebraic sets, then $\bigcap Y_\alpha = Z(\bigcup T_\alpha)$, so $\bigcap Y_\alpha$ is also an algebraic set. Finally, the empty set $\varnothing = Z(1)$, and the whole space $\mathbf{A}^n = Z(0)$.

Definition. We define the *Zariski topology* on $\mathbf{A}^n$ by taking the open subsets to be the complements of the algebraic sets. This is a topology, because according to the proposition, the intersection of two open sets is open, and the union of any family of open sets is open. Furthermore, the empty set and the whole space are both open.

Example 1.1.1. Let us consider the Zariski topology on the affine line $\mathbf{A}^1$. Every ideal in $A = k[x]$ is principal, so every algebraic set is the set of zeros of a single polynomial. Since k is algebraically closed, every nonzero polynomial $f(x)$ can be written $f(x) = c(x - a_1) \cdots (x - a_n)$ with $c,a_1, \ldots ,a_n \in k$. Then $Z(f) = \{a_1, \ldots ,a_n\}$. Thus the algebraic sets in $\mathbf{A}^1$ are just the finite subsets (including the empty set) and the whole space (corresponding to $f = 0$). Thus the open sets are the empty set and the complements of finite subsets. Notice in particular that this topology is not Hausdorff.

Definition. A nonempty subset Y of a topological space X is *irreducible* if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y. The empty set is not considered to be irreducible.

Example 1.1.2. $\mathbf{A}^1$ is irreducible, because its only proper closed subsets are finite, yet it is infinite (because k is algebraically closed, hence infinite).

Example 1.1.3. Any nonempty open subset of an irreducible space is irreducible and dense.

Example 1.1.4. If Y is an irreducible subset of X, then its closure $\bar{Y}$ in X is also irreducible.

Definition. An *affine algebraic variety* (or simply *affine variety*) is an irreducible closed subset of $\mathbf{A}^n$ (with the induced topology). An open subset of an affine variety is a *quasi-affine variety*.

These affine and quasi-affine varieties are our first objects of study. But before we can go further, in fact before we can even give any interesting examples, we need to explore the relationship between subsets of $\mathbf{A}^n$ and ideals in A more deeply. So for any subset $Y \subseteq \mathbf{A}^n$, let us define the *ideal* of Y in A by

$$I(Y) = \{f \in A \mid f(P) = 0 \text{ for all } P \in Y\}.$$

Now we have a function Z which maps subsets of A to algebraic sets, and a function I which maps subsets of $\mathbf{A}^n$ to ideals. Their properties are summarized in the following proposition.

Proposition 1.2.

(a) *If $T_1 \subseteq T_2$ are subsets of A, then $Z(T_1) \supseteq Z(T_2)$.*
(b) *If $Y_1 \subseteq Y_2$ are subsets of $\mathbf{A}^n$, then $I(Y_1) \supseteq I(Y_2)$.*
(c) *For any two subsets Y_1, Y_2 of $\mathbf{A}^n$, we have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.*
(d) *For any ideal $\mathfrak{a} \subseteq A$, $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, the radical of $\mathfrak{a}$.*
(e) *For any subset $Y \subseteq \mathbf{A}^n$, $Z(I(Y)) = \bar{Y}$, the closure of Y.*

PROOF. (a), (b) and (c) are obvious. (d) is a direct consequence of Hilbert's Nullstellensatz, stated below, since the radical of $\mathfrak{a}$ is defined as

$$\sqrt{\mathfrak{a}} = \{f \in A \mid f^r \in \mathfrak{a} \text{ for some } r > 0\}.$$

To prove (e), we note that $Y \subseteq Z(I(Y))$, which is a closed set, so clearly $\bar{Y} \subseteq Z(I(Y))$. On the other hand, let W be any closed set containing Y. Then $W = Z(\mathfrak{a})$ for some ideal $\mathfrak{a}$. So $Z(\mathfrak{a}) \supseteq Y$, and by (b), $IZ(\mathfrak{a}) \subseteq I(Y)$. But certainly $\mathfrak{a} \subseteq IZ(\mathfrak{a})$, so by (a) we have $W = Z(\mathfrak{a}) \supseteq ZI(Y)$. Thus $ZI(Y) = \bar{Y}$.

Theorem 1.3A (Hilbert's Nullstellensatz). *Let k be an algebraically closed field, let $\mathfrak{a}$ be an ideal in $A = k[x_1, \ldots, x_n]$, and let $f \in A$ be a polynomial which vanishes at all points of $Z(\mathfrak{a})$. Then $f^r \in \mathfrak{a}$ for some integer $r > 0$.*

PROOF. Lang [2, p. 256] or Atiyah–Macdonald [1, p. 85] or Zariski–Samuel [1, vol. 2, p. 164].

Corollary 1.4. *There is a one-to-one inclusion-reversing correspondence between algebraic sets in $\mathbf{A}^n$ and radical ideals* (i.e., *ideals which are equal to their own radical*) *in A, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. Furthermore, an algebraic set is irreducible if and only if its ideal is a prime ideal.*

PROOF. Only the last part is new. If Y is irreducible, we show that $I(Y)$ is prime. Indeed, if $fg \in I(Y)$, then $Y \subseteq Z(fg) = Z(f) \cup Z(g)$. Thus $Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$, both being closed subsets of Y. Since Y is irreducible, we have either $Y = Y \cap Z(f)$, in which case $Y \subseteq Z(f)$, or $Y \subseteq Z(g)$. Hence either $f \in I(Y)$ or $g \in I(Y)$.

Conversely, let $\mathfrak{p}$ be a prime ideal, and suppose that $Z(\mathfrak{p}) = Y_1 \cup Y_2$. Then $\mathfrak{p} = I(Y_1) \cap I(Y_2)$, so either $\mathfrak{p} = I(Y_1)$ or $\mathfrak{p} = I(Y_2)$. Thus $Z(\mathfrak{p}) = Y_1$ or Y_2, hence it is irreducible.

Example 1.4.1. $\mathbf{A}^n$ is irreducible, since it corresponds to the zero ideal in A, which is prime.

Example 1.4.2. Let f be an irreducible polynomial in $A = k[x,y]$. Then f generates a prime ideal in A, since A is a unique factorization domain, so the zero set $Y = Z(f)$ is irreducible. We call it the *affine curve* defined by the equation $f(x,y) = 0$. If f has degree d, we say that Y is a curve of *degree d*.

Example 1.4.3. More generally, if f is an irreducible polynomial in $A = k[x_1, \ldots, x_n]$, we obtain an affine variety $Y = Z(f)$, which is called a *surface* if $n = 3$, or a *hypersurface* if $n > 3$.

Example 1.4.4. A maximal ideal $\mathfrak{m}$ of $A = k[x_1, \ldots, x_n]$ corresponds to a minimal irreducible closed subset of $\mathbf{A}^n$, which must be a point, say $P = (a_1, \ldots, a_n)$. This shows that every maximal ideal of A is of the form $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$, for some $a_1, \ldots, a_n \in k$.

Example 1.4.5. If k is not algebraically closed, these results do not hold. For example, if $k = \mathbf{R}$, the curve $x^2 + y^2 + 1 = 0$ in $\mathbf{A}^2_{\mathbf{R}}$ has no points. So (1.2d) is false. See also (Ex. 1.12).

Definition. If $Y \subseteq \mathbf{A}^n$ is an affine algebraic set, we define the *affine coordinate ring* $A(Y)$ of Y, to be $A/I(Y)$.

Remark 1.4.6. If Y is an affine variety, then $A(Y)$ is an integral domain. Furthermore, $A(Y)$ is a finitely generated k-algebra. Conversely, any finitely generated k-algebra B which is a domain is the affine coordinate ring of some affine variety. Indeed, write B as the quotient of a polynomial ring $A = k[x_1, \ldots, x_n]$ by an ideal $\mathfrak{a}$, and let $Y = Z(\mathfrak{a})$.

Next we will study the topology of our varieties. To do so we introduce an important class of topological spaces which includes all varieties.

Definition. A topological space X is called *noetherian* if it satisfies the *descending chain condition* for closed subsets: for any sequence $Y_1 \supseteq Y_2 \supseteq \ldots$ of closed subsets, there is an integer r such that $Y_r = Y_{r+1} = \ldots$.

Example 1.4.7. $\mathbf{A}^n$ is a noetherian topological space. Indeed, if $Y_1 \supseteq Y_2 \supseteq \ldots$ is a descending chain of closed subsets, then $I(Y_1) \subseteq I(Y_2) \subseteq \ldots$ is an ascending chain of ideals in $A = k[x_1, \ldots, x_n]$. Since A is a noetherian ring, this chain of ideals is eventually stationary. But for each i, $Y_i = Z(I(Y_i))$, so the chain Y_i is also stationary.

Proposition 1.5. *In a noetherian topological space X, every nonempty closed subset Y can be expressed as a finite union $Y = Y_1 \cup \ldots \cup Y_r$ of irreducible closed subsets Y_i. If we require that $Y_i \not\supseteq Y_j$ for $i \neq j$, then the Y_i are uniquely determined. They are called the* irreducible components *of Y.*

PROOF. First we show the existence of such a representation of Y. Let $\mathfrak{S}$ be the set of nonempty closed subsets of X which *cannot* be written as a finite union of irreducible closed subsets. If $\mathfrak{S}$ is nonempty, then since X is noetherian, it must contain a minimal element, say Y. Then Y is not irreducible, by construction of $\mathfrak{S}$. Thus we can write $Y = Y' \cup Y''$, where Y' and Y'' are proper closed subsets of Y. By minimality of Y, each of Y' and Y'' can be expressed as a finite union of closed irreducible subsets, hence Y also, which is a contradiction. We conclude that every closed set Y can be written as a union $Y = Y_1 \cup \ldots \cup Y_r$ of irreducible subsets. By throwing away a few if necessary, we may assume $Y_i \not\supseteq Y_j$ for $i \neq j$.

Now suppose $Y = Y'_1 \cup \ldots \cup Y'_s$ is another such representation. Then $Y'_1 \subseteq Y = Y_1 \cup \ldots \cup Y_r$, so $Y'_1 = \bigcup (Y'_1 \cap Y_i)$. But Y'_1 is irreducible, so $Y'_1 \subseteq Y_i$ for some i, say $i = 1$. Similarly, $Y_1 \subseteq Y'_j$ for some j. Then $Y'_1 \subseteq Y'_j$, so $j = 1$, and we find that $Y_1 = Y'_1$. Now let $Z = (Y - Y_1)^-$. Then $Z = Y_2 \cup \ldots \cup Y_r$ and also $Z = Y'_2 \cup \ldots \cup Y'_s$. So proceeding by induction on r, we obtain the uniqueness of the Y_i.

Corollary 1.6. *Every algebraic set in $\mathbf{A}^n$ can be expressed uniquely as a union of varieties, no one containing another.*

Definition. If X is a topological space, we define the *dimension* of X (denoted $\dim X$) to be the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of distinct irreducible closed subsets of X. We define the *dimension* of an affine or quasi-affine variety to be its dimension as a topological space.

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