Mathematical Methods in Finance and Economics

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PREFACE

The increasing reliance of finance and economics on mathematical models has made it necessary for students to have some understanding of certain mathematical methods. However, the highly technical nature of many of the texts in applied mathematics has often discouraged students from using quantitative methods as tools in conceptualizing and solving business problems.

This text attempts to introduce the student to what we consider to be essential mathematical background for managers and economists, without the burden of unnecessary technical details. Therefore, the emphasis is almost entirely on "how to" rather than on mathematical proofs. Our experience has shown that teaching by means of examples is the most effective way of communicating this material to students.

It is not expected that the students using this text have an extensive background in mathematics: however, we do require introductory courses in elementary calculus and statistics. Therefore, we believe that this text is most suitable for juniors or seniors in undergraduate business and economics programs, as well as graduate students in the same areas.

Regarding the structure of the book, we note that much of Chapter 3 (and especially Section 3.5) is necessary to the reading of Chapter 4. Chapter 5 is essentially independent of Chapter 4, except where it uses linear programming methods to solve matrix games. Chapter 6 requires more mathematical sophistication on the part of the reader than the other chapters, and it may be beyond the reach of some students; however, we have attempted to keep its level of mathematical difficulty to an absolute minimum.

The problems after each chapter have been designed not only to illustrate the methods developed in the text, but also to introduce other applications.

It should be observed that in Chapters 3–6 (except Section 6.5), we have generally used capital letters to denote matrices and vectors. There are two exceptions to this rule: first, capital letters K and P have been used throughout to denote the cost and profit objective functions in linear programs; second, owercase letters p and q have been used to denote probability vectors in Chapter 5.

We suggest that teachers using this text try to cover the topics in the sequence in which they are presented. With the exception of the interdependencies noted above, the chapters can be covered independently of one another.

We hope that our treatment of this material will prove useful not only to university students of business and economics, but also to those outside the academic world.

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CONTENTS

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THE NATURE OF MATHEMATICAL FINANCE AND ECONOMICS

The application of mathematics to financial theory and economic theory is pervasive. Mathematics is utilized to simplify presentation, summarize and clarify concepts, and help integrate complex issues and relationships into a systematic body of knowledge.

Before treating the various mathematical concepts, we shall provide a perspective on finance and economics that will serve as a quick review of the fundamentals of both disciplines and be a guide to the areas that can be treated mathematically by means of various mathematical models and techniques.

1.1 MATHEMATICAL MODELS

A mathematical model for finance or economics is an abstraction of the complexities of the real world designed to facilitate analysis of real problems through a system of functional relationships among a set of variables.

Mathematical models (the most abstract models of all) can be classified into two categories: computational models, based on a mathematical structure yielding a general explicit solution or a range of solutions, and analytic models, which yield general optimality or equilibrium conditions.

Both categories of mathematical models will be discussed in this text.

1.2 INGREDIENTS OF MATHEMATICAL MODELS

The development of a mathematical model requires a systematic approach of the following order:

1. Problem identification and definition; hypothesis formulation.
2. Statement of the assumptions of the model.

3. Identification of relevant variables based on understanding of the relevant environment and the nature of the problem with reference to the up-to-date literature on the subject under study. Variables are categorized as endogenous (inside the system or determined by the model) or exogenous (outside the system—not determined by the model), and as dependent or independent (controllable or uncontrollable) variables.
4. Development of the model; that is, the establishment of the mathematical relationships among the variables.
5. Solution to the model.
6. Testing (using various techniques) the validity of the equations derived from the model.
7. Conclusions.[1]

Although the first four steps will be dealt with to some degree, beginning in Chapter 2 with the net present value (NPV) and the internal rate of return (IRR) models, the main emphasis will be on mathematical techniques for solving models.

1.3 THE THEORY OF FINANCE

Definition 1.3.1. The theory of finance is the study of the accumulation and the allocation of resources over time and at a period in time under varied states of the world. The theory of finance also seeks to explain how the allocation of resources is facilitated by the existence of money and capital markets.

The theory of finance is concerned with the determination of the value of the firm as a going concern, the identification and analysis of factors with direct and indirect influence on this value, and with the valuation of investment opportunities. It deals with the acquisition and optimal application of funds (i.e., the distribution of assets among competing investments) and the effects of these funds on the profitability and growth of the business firm or on the efficiency of resource utilization by nonprofit institutions.

The economic value of the firm as a going concern is, as will be shown in detail in Chapter 2, the discounted value of streams of income that the firm will generate in the future. Since the corporate firm has an indefinite life, its market value, can be approximated by the following model:

$$V = \sum_{t=1}^{\infty} E_t \Big/ \prod_{n=1}^{t} (1+K_n) \tag{1.3.1}$$

[1] N. Paul Loomba, *Management—A Quantitative Perspective* (New York: Macmillan, 1978), pp. 42–49.

If all the K_n's are equal,

$$V = \sum_{t=1}^{\infty} E_t \Big/ (1+K)^t \tag{1.3.2}$$

or, assuming continuous discounting,

$$V = \int_0^{\infty} E_t e^{-Kt}\, dt \tag{1.3.3}$$

where

V = market value of the firm

K = an appropriate discount factor

t = time period

E_t = cash earnings of the firm in period t

Optimization models focus on the variable V, on the expected stream of income and its determinants and on the determinants of the discount factor K.

A positive and significant correlation between profits and firm value has been confirmed repeatedly. Hence, we shall assume that the objective of the management of a business entity is the maximization of profits. The profit function is ordinarily of the form

$$E_t = \left(P_t Q_t - \sum_{i}^{n} f_i X_i - \mathrm{FC} - i_t B_t \right)(1-T) \tag{1.3.4}$$

where

E = earnings of firm after interest and taxes

P = price of output

Q = number of units sold

f_i = price of one unit of input i

X_i = total units of input i used in the production of output

n = total number of inputs used in production

FC = fixed cost

i = average interest rate paid by the firm on borrowed funds

B = total borrowing (leverage) incurred by the firm. The value of funds owed to nonowners

T = corporate tax rate

The maximization of this profit function is equivalent to the maximization of the wealth of the stockholders, the owners of the firm.

Profit maximization of this kind, however, is not common. The firm is subject to many constraints that limit its profit potential. These constraints can be technical, resource related, financial, environmental, and political in nature. Many of these constraints are quantifiable and can readily be incorporated in a profit maximization model, as will be demonstrated in Chapters 4 and 6. Constraints that cannot be monetized will be ignored throughout the text. A typical constrained profit maximization model is of the form

$$\text{Maximize} \quad E_t = \left(P_t Q_t - \sum_i^n f_i X_i - \text{FC} - i_t B_t \right)(1-T) \tag{1.3.5}$$

$$\text{subject to} \quad \sum \alpha_i X_i = \bar{S} + \bar{B}$$

$$\sum \alpha_i = 1, \qquad X_i \geq 0$$

where

α_i = percent of total capital expended on input i

The constraint here is that there is a fixed amount of equity ($\bar{S}$) and debt capital ($\bar{B}$) that the firm may utilize in the procurement of inputs X_i. If $\bar{S}$ and $\bar{B}$ represent a ceiling on equity and debt capital, the constraint will then become

$$\sum \alpha_i X_i \leq \bar{S} + \bar{B}$$

There are various ways of solving the models above. The mathematical techniques would be substantially different depending on whether the constraints had an equality sign or an inequality sign. Chapters 4 and 6 will address these and other issues relating to models of this type and more complex ones.

Although the overwhelming weight of the empirical evidence supports the profit maximization hypothesis, many new theories are making inroads in the finance literature. The sales maximization school fathered by William Baumol of Princeton University contends that managers do not maximize profits but do maximize their own utility, which is a function of the size of the firm they run, the size of their office, their salary and fringe benefits, etc. The other school of thought that is gaining many new adherents is the behavioral school, which looks at the firm as a collection of competing interest groups, the most powerful and convincing of which will end up setting the policies of the firm. The role of management will be one of reducing friction and ensuring harmony among the various groups within the organization. Neither the sales maximization school nor the behavioral school will be expressly treated in this text, although some of the mathematical methods advanced in the chapters to follow are very useful in the presentation and analysis of such theories.

So much for the corporate entity; the behavior of individuals as economic agents will also be dealt with in many parts of this text. Individuals in the marketplace behave so as to maximize utility,[2] that is, to maximize the satisfaction derived from an insatiable consumption of goods. Utility maximization is invariably subject to constraints, the most important of which is income. A simplified model assuming two consumption goods X_1 and X_2 follows:

$$\text{Max } U = f(X_1, X_2) \tag{1.3.6}$$

Subject to

$$X_1 \geqq 0,\ X_2 \geqq 0$$

$$P_1 X_1 + P_2 X_2 = Y$$

where

U = utility

X_1 = number of units consumed of good X_1

X_2 = number of units consumed of good X_2

P_1, P_2 = prices of good 1 and good 2, respectively

Y = income

The maximization of this model will yield demand functions by consumers for both goods X_1 and X_2:

$$X_1 = f(P_1, Y) \qquad \text{and} \qquad X_2 = f(P_2, Y)$$

with

$$\frac{\partial X_i}{\partial P_i} < 0, \qquad \frac{\partial X_i}{\partial Y} > 0$$

1.4 THE THEORY OF FINANCE—A CLOSER LOOK

An easy (but not fully comprehensive) way to introduce the specific areas with which the theory of finance deals is through the examination of the balance sheet and income statement items of a business enterprise, the JAK Corporation (see Table 1.1).

The total assets of the JAK Corporation, \$19,400,000 in Table 1.1, represent the total investment that has been made in the firm. The distribution of these assets shows how the firm's management chose to commit its resources with the objective of generating maximum income

[2]Utility is a measure of happiness (psychic gain) derived from making choices in the marketplace. Utility theory is a normative model defining "rational behavior" for the participants in a social economy. If decision makers behaved in accordance with the axioms of cardinal (measurable) utility (comparability, transitivity, strong independence, measurability, and ranking) then their behavior under conditions of uncertainty becomes predictable. The reader is advised to consult E. F. Fama and M. H. Miller, *The Theory of Finance* (New York: Holt, Rinehart and Winston, 1972), Chapter 5.

TABLE 1.1. JAK Corporation Balance Sheet, December 31, 1980

Account		Dollar Value
ASSETS		
Current Assets		
1. Cash		900,000
2. Marketable securities at cost (market value $1,890,000)		1,700,000
3. Accounts receivable	$4,200,000	
Less allowance for bad debt	(200,000)	4,000,000
4. Inventories		5,400,000
Total current assets		12,000,000
5. Fixed Assets		
Land		900,000
Building	7,600,000	
Machinery	1,900,000	
Office equipment	200,000	
Less accumulated depreciation	(3,600,000)	6,100,000
Total fixed assets		7,000,000
Other Assets: Prepayments and deferred charges		200,000
6. Intangibles (goodwill, patents, trademarks)		200,000
Total Assets		19,400,000

streams that will maximize the economic value of the firm. The total asset value is their book value, a historical value that is not of great concern to financial theory. The real concern of financial theory is the ability of each of the assets to generate income capable of affecting the profitability of the firm and, hence, its market value.

The optimal quantity of cash balance (item 1, Table 1.1) has necessarily been of great concern to financial theorists since the early 1950s. Economic order quantity (EOQ) models were utilized for the determination of the optimal quantity of cash to be held on hand, by means of deterministic or stochastic models. These models, in their simplest form, use unconstrained total cost minimization (total cost function equals the cost of carrying the inventory and the cost of placing inventory orders) to derive a formula for cash management.

The theory of investment, a subdiscipline of the theory of finance, deals with methods used to determine the optimal portfolio of securities (item 2, Table 1.1); that is, methods used to determine the collection of securities that will minimize portfolio risk (measured by the standard deviation of

TABLE 1.1 *(continued)*

Account	Dollar Value
LIABILITIES	
Current Liabilities	
7. Accounts payable	2,000,000
8. Notes payable	1,700,000
Accrued expenses payable	660,000
Federal income tax	640,000
Total current liabilities	5,000,000
Long-Term Liabilities	
First mortgage bonds;	
5% interest due 1985	5,400,000
Total liabilities	10,400,000
STOCKHOLDERS' EQUITY	
Capital Stock	
Preferred stock, 5% cumulative, $100 par value each; authorized, issued, and outstanding 12,000 shares	1,200.000
9. Common stock, $5 par value each authorized issued, and outstanding 300,000 shares	3,000,000
10. Capital surplus	1,400,000
11. Accumulated retained earnings	3,400,000
Total stockholders' equity	9,000,000
Total Liabilities and Stockholders' Equity	19,400,000

portfolio rate of return) given a rate of return, or maximize the portfolio rate of return given a level of risk. The optimal portfolio, therefore, takes risk and returns on assets into consideration and determines the best combination of these assets. Which marketable securities are chosen and how many of each the firm should hold can be determined using techniques developed within the theory of investment, which will be introduced in Chapter 2.

Financial theory can also be applied to item 3 in Table 1.1. The analysis of the composition of accounts receivable (aging schedules and risk profiles of the debtors) and of various ways to enhance the collection of receivables, both physical (e.g., location of collection centers) and financial (discounts granted by the firm to its customers for early payments, and the relationship of credit policies to sales and profits, etc.) is well within the domain of the theory of finance.

The optimal level of inventory (item 4, Table 1.1), as well as the composition of inventory, is arrived at by using EOQ models of the deterministic as well as the stochastic kind. These models are similar to those utilized in the management of cash, both conceptually and mathematically.

Fixed (long-term) assets (item 5, Table 1.1) are very much the concern of the theory of finance. To a great extent, these assets determine the productive capacity of the firm. Each component of these assets can be determined by means of capital budgeting models. Decision-making models for the acquisition and the disposition of fixed assets and methods of deciding on the optimal combination of these assets (i.e., looking at fixed assets from a portfolio perspective) will be discussed extensively in Chapter 2.

Being historical in perspective, the balance sheet describes the structure of assets that the firm considers most effective in the pursuit of the profit goal. The task of the financial theorist in this regard is, therefore, to decide which combination of assets is optimal, and whether the firm's present capital structure is indeed optimal.

The theory of finance is thus concerned with the liabilities and stockholders' equity side of the balance sheet. These represent total money capital employed in the firm. The distribution of this capital between equity and debt capital and its effects on the total value of the firm are important topics for financial theorists.

For example, the concern with accounts payable (item 7, Table 1.1) is with their maturity, adequate synchronization with cash receipts, and strategy as to the firm's payment schedule (pay now or pay later), all of which have a direct impact on the profitability of the firm. Notes payable (item 8, Table 1.1) are short-term borrowings used mostly to finance short-term assets and particularly to eliminate cash shortages. When to borrow, how much, and through what means (private placement[3] versus public offering) are all issues of concern.

Federal income tax liabilities are also quite important. A dollar of taxes saved is a dollar earned. Tax obligations are costs, and the smaller they are, the larger the profits after taxes will be; this is important because the firm maximizes long-run profit after taxes.

Long-term liabilities, otherwise known as leverage, are a major concern. The impact of the borrowing decision on the total value of the firm is subject to much controversy. What is not debatable, however, is that with taxes included, leverage does have an impact on the value of the firm.

Furthermore, the theory of finance also deals with

the level and the impact of leverage (borrowing) on the profitability of the firm under different conditions of the world, and/or the ability of the firm to finance its assets in the future;

[3]Private placement is the selling of a security issue to one firm (person) or to a group of firms (persons). No public announcement and no registration with the Securities and Exchange Commission are required.