THE FINITE ELEMENT METHOD

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Fundamentals and Applications

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PREFACE

Although the finite element method originated in structural mechanics, its roots belong in mathematics since it is a particular class of approximation procedure. In this book, the finite element method is presented not as it developed historically but within the framework of a general taxonomy.

In the opening chapter, the formulation and classification of physical problems is considered. This is followed by a review of field or continuum problems and their approximate solutions by the method of trial functions. It is shown that the finite element method is a subclass of the method of trial functions, and further, that a *finite element formulation* can, in principle, be developed for most trial function procedures. Variational and residual trial function methods are considered in some detail and their convergence is examined. After a review of the calculus of variations, both in classical and Hilbert space form, the fundamentals of the finite element method are introduced. A classification is also presented for the various categories of the finite element method. Convergence is investigated at some length. To illustrate the variational approach, the Ritz finite element method is then outlined, both for an equilibrium problem using the classical calculus of variations and for equilibrium and eigenvalue problems using the Hilbert space approach. The application of the finite element method to solid and structural mechanics follows, although no attempt has been made to provide other than a basic introduction to these areas since excellent coverage is available in standard texts. Applications to other physical problems are considered in the chapters pertaining to the Laplace, Helmholtz, wave, and diffusion equations, as well as in succeeding chapters. An extensive list of additional references is also given. The aim of this book has been to demonstrate the generality of the finite element method by providing a unified treatment of fundamentals and a broad coverage of applications.

An advanced knowledge of mathematics is not required for this book, since only a reasonable acquaintance with differential and integral calculus has been presupposed. Matrix algebra and calculus are used extensively, and are reviewed in the appendices for those unfamiliar with these subjects. In Chapters 4 and 7, concepts from functional analysis are introduced. While the Hilbert space approach given in these sections allows a powerful generalization of variational and finite element methods which should not be overlooked, these chapters can be omitted on a first reading.

By appropriate selection of chapters, this book may be found suitable for undergraduate and graduate courses. The authors intended it to appeal not only to engineers and others concerned with practical applications, but also to scientists and applied mathematicians.

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CHAPTER

1

THE FORMULATION OF PHYSICAL PROBLEMS

1.1 INTRODUCTION

The concern of the engineer, the scientist, and the applied mathematician is with *physical phenomena*, although from different points of view. To define and solve a *physical problem*, the *state* or *configuration* of the phenomenon must be described at one or more times. The entirety of the phenomena being considered constitutes the *system*, and its state is described by the *physical quantities* by which it is manifested. In a given problem, some of these quantities may be prescribed or otherwise fixed, while others are unknown or variable and constitute the *variables* or *parameters* of the problem. That set of variables which is the minimum number needed to reference (or to determine) the state of the system completely is known as the set of *independent variables*. All other variables describing the system will be dependent on this set, and are thus *dependent variables*. In many systems, there are specified or physically imposed conditions known as *constraints* which reduce the number of independent variables from that which would otherwise be required. For example, a rotating flywheel is constrained by its bearings so that there are only two independent variables in this system (e.g., the angle from the initial position and time).

If time, when present, is deleted from a set of physical quantities, the remainder are called a set of *generalized coordinates* of the system. In any physical system, the maximum number of independent generalized coordinates (i.e., those that can be varied arbitrarily and independently without violating any of the constraints) is known as the *number of degrees of freedom* of the system.

A *holonomic* system can always be described by a set of generalized coordinates that are independent. A *nonholonomic* system requires a set of generalized coordinates not all of which are independent, being related by *equations of constraint*. In either case, time, if present, must also be added to complete the description. The number of degrees of freedom is always that number of generalized coordinates that can be regarded as independent, being the full number of coordinates in the case of a holonomic system, and the full number minus the number of constraint equations if the system is nonholonomic [1–4]. If the constraints are implicitly taken into account in the formulation of the problem, there is no need for the separate equations of constraint, and what would otherwise be a nonholonomic problem becomes a holonomic one.

If a problem involves a system of discrete interconnected elements, then the phenomenon may be described by a finite number of degrees of freedom, whereas the description of a phenomenon in a continuum requires a knowledge of quantities at every point so that a continuum problem has an infinite number of degrees of freedom. The former is known as a *discrete* (or *lumped-parameter*) system, while the latter is a *continuous* system. Primarily this book considers problems of the continuous type (often called *field problems*), although in the latter part of Chapter 8 a discrete system of interconnected structural members is considered. Continuous problems are often approximated as discrete problems, and it will be shown that the *finite element* method is a means of accomplishing this transformation and solving the resulting set of equations.

1.2 CLASSIFICATION OF PHYSICAL PROBLEMS

Most problems in engineering, physics, and applied mathematics can be classified as either *discrete* or *continuous*. A discrete system consists of a finite number of interconnected elements, whereas a continuous system involves a phenomenon over a continuous region. Several masses interconnected by a system of springs is an example of the former, and heat conduction in a block an example of the latter. It should be noted here in connection with a discrete system, that the term *variable* is used in a singular sense to mean a separate quantity (e.g., the Cartesian coordinates x_1, x_2 of two masses would each be a variable). In a continuous system, *variable* is used in a plural sense to mean any one of an allowable set of similar quantities (e.g., the variable x might be used for the x Cartesian coordinate of each one of the points in the region considered). The variable *time* is an exception in discrete systems, where it is normally used in the plural sense.

Discrete and continuous systems can each be further subdivided into *equilibrium*, *eigenvalue*, and *propagation* problems:

a. *Equilibrium problems* are those in which the system state remains constant with time, and are often known as steady-state problems. Examples are the statics of structures, steady compressible flow, stationary electrostatic fields, and steady voltage distributions in networks.

b. *Eigenvalue problems* can be considered as extensions of equilibrium problems in which, in addition to the corresponding steady-state configuration, specific or critical values of certain other parameters must be determined as well. Examples in this category include the buckling and stability of structures, natural frequency problems in mechanical systems, and the determination of resonances in electrical circuits.

c. *Propagation problems* include transient and unsteady-state phenomena, and are those in which a subsequent state of the system has to be related to an initially known state. Stress waves in elastic continua, the development of self-excited vibrations, and unsteady heat conduction are examples of propagation problems.

1.3 CLASSIFICATION OF THE EQUATIONS OF A SYSTEM

In a physical problem, whether discrete or continuous, the state of the system can be described by variables, of which a set $x_1, x_2, \ldots, x_n$ (collectively represented by x_j) is independent and a set $u_1, u_2, \ldots, u_m$ (collectively represented by u_i) is dependent. The *region* of the system is defined by the sets of all possible values that the x_j can have. A particular set of allowable values of x_j defines a *point* in the region. If at a point (with the remaining independent variables held constant), one of the x_j can either be increased or decreased to another allowable value, the point is said to be in the *interior* of the region. If the variable can be decreased to another allowable value but an increase gives a value outside the prescribed range, or vice versa, then the point is on the ***boundary*** of the region. If the boundary points are deleted from the region, the remaining (interior) points constitute the *domain*. Sometimes there is no (upper and/or lower) bound on one or more of the independent variables, and in this case the boundary is said to be *open* with respect to that variable. When all the independent variables are bounded, the boundary is *closed*. In some cases, the region is internally subdivided by *interior boundaries*.

From physical laws (and often also from prescribed conditions), various relationships will be deducible in the domain $\mathscr{D}$ between the dependent and independent variables, and the *domain* or *field equations* will thus be one or more equations of the form

$$f_{\mathscr{D}}(u_1, u_2, \ldots, u_m; x_1, x_2, \ldots, x_n) = 0 \quad \text{in} \quad \mathscr{D}. \tag{1.1}$$

In addition, there will be one or more equations (the *boundary conditions*) applying over the bounding surface $\mathscr{S}$, of the form

$$f_{\mathscr{S}}(u_1, u_2, \ldots, u_m; x_1, x_2, \ldots, x_n) = 0 \quad \text{in} \quad \mathscr{S}. \tag{1.2}$$

Equations (1.1) and (1.2) are the *governing equations* or *governing relations* of the system. It is to be understood in Eqs. (1.1) and (1.2) that not every variable need occur in each equation, and that the functions $f_{\mathscr{D}}$ and $f_{\mathscr{S}}$ include algebraic, differential, and integral operations on the variables. In general, the u_i occurring in Eqs. (1.1) and (1.2) will not be the full set of all possible dependent variables, but some subset of these. The variables u_i and x_j need not be restricted to scalars, but can be vectors or matrices.

In discrete or lumped-parameter systems [5], some or all of the independent variables are often set equal to constants. In the latter case, the region collapses to a point and there is no boundary and hence no boundary conditions. For example, in the equilibrium problem of a mass hanging on a spring, the independent variables (mass and characteristic spring rate) are set equal to constants and the dependent variable (the position of the mass) is then determined from the (condensed) domain equation.

A problem will be considered to be *well behaved*† if there are sufficient equations (1.1) and (1.2) so that solutions for those u_i occurring in Eqs. (1.1) and (1.2) not only exist but are also unique. Explicit solutions of the u_i will be sought of the general form

$$u_i = f_i(x_1, x_2, \ldots, x_n). \tag{1.3}$$

† See Crandall [5]; also the concept of *well posed* in Ames [6], Hadamard [7], and Courant and Hilbert [8].

It will be noted that the definition of domain used in this section agrees with that of set and function theory, where the *domain of a function* is the set on which the function is defined, and the *range or image* is the set of values assumed by the function. For the function $f_i(x_1, x_2, \ldots, x_n)$ on the right-hand side of Eq. (1.3) (which is valid for the interior of the region), the domain in the mathematical sense can consist of all allowable values of $x_1, x_2, \ldots, x_n$, which agrees with the earlier definition. Since the domain of the function $f_i(x_1, x_2, \ldots, x_n)$ relates to the solution u_i, it is sometimes called the *solution domain* of the problem.

The various classes of problem discussed earlier (discrete, continuous, equilibrium, eigenvalue, propagation) have the different types of governing equations (1.1) and (1.2) shown in Table 1.1. It should be noted when using this table that *simultaneous* means *to be considered simultaneously* and can refer to sets of nonlinear as well as linear equations. *Initial conditions* are domain conditions that are specified at an initial time.

It will be seen from Table 1.1 that equation sets for discrete systems are simpler to deal with than those for continuous systems. Many of the approximate methods of solution for continuous systems reduce the number of degrees of freedom of the system from infinity to a finite number, and thus reduce the problem to the simpler one of a discrete system. The finite element method is one such approximation method.

TABLE 1.1

Relationships between Problem Types and Corresponding Sets of Governing Equations

Problem classification	Governing equations of the problem	
	Discrete	Continuous
Equilibrium	(Simultaneous) algebraic equations	Ordinary or partial differential equations with closed boundary conditions
Eigenvalue	(Simultaneous) algebraic equations or ordinary differential equations reducible to algebraic equations	Ordinary or partial differential equations with closed boundary conditions
Propagation	(Simultaneous) ordinary differential equations with prescribed initial conditions	Partial differential equations with prescribed initial conditions and open boundary conditions

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