CONSTRUCTIVE FORMALISM

ESSAYS ON THE FOUNDATIONS OF MATHEMATICS

BY

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TO

MY WIFE.

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PREFACE

The main forces which have shaped the foundations of mathematics over the past twenty-five years have been antithetical in purpose but complementary in effect. To Hilbert's formalism we owe the detailed analysis of the structure of mathematical systems and the imaginative conception of mathematics as its own object of discourse; to the constructivism of Brouwer, the critique of classical logic and the intuitive notion of a finitist proof.

The formalist-finitist controversy in the foundations of mathematics was resolved, in principle, by Wittgenstein's analysis of the chacteristics of a formal language. Wittgenstein showed that in a formal language the *meaning* of the signs is a purely *functional* property of the language; it follows that Brouwer's denial of the *validity* of a formal axiom—the *tertium non datur*—was totally mistaken. The conclusion to be drawn from the finitist critique is not that certain parts of mathematics are incorrect but that the currently accepted interpretation of the signs, in particular the interpretation of the quantifiers A and E in terms of universality and existence is *untenable*. One cannot dispute a formal equivalence like

$$\sim Ax(P(x)) = Ex(\sim P(x))$$

but one may well be able to show that the use of the quantifiers A and E in the formula is not consistent with the ordinary usage of the terms "for all" and "there exists", so that in a system in which this formula holds, "A" and "E" are not synonymous with the universal and existential operators.

Constructivism and formalism found a point of contact in *recursive number theory* which was developed by Skolem, and by Herbrand and Gödel in their construction of non-demonstrable propositions. Recursive number theory plays a fundamental part in the fusion of these two modes of thought in the present work.

The aim of *constructive formalism* is to replace the intuitive notion of a finitist proof by the strictly formal property of demonstrability in a formal system. This is accomplished by the construction of a mathematical system—the equation calculus—which operates independently of the axioms and constants of logic. This system affords a means of proving certain types of logical formulæ and consequently effects a reduction of logic to mathematics.

The necessity for some equivalent of the theory of types involves any system founded upon the concept of class in intolerable complications, but even apart from questions of expediency there are good grounds for denying the class concept a primary part in a mathematical system. The equation calculus gives to *function* the fundamental role that classical analysis assigns to related classes. A function is defined by the introductory equations of its sign, which by means of the transformation rules of the calculus, serve to transform the function-sign into a definite numeral, when definite numerals are assigned to the argument places in the function-sign.

Though the reduction of mathematics to a formal system independent of logic solves many of the problems in the foundations of mathematics —in the sense that it eliminates them—there remains the 'ultimate' problem of the relation of mathematics to reality, which finds expression in such questions as: "If mathematics is a purely formal system how is the application of mathematics possible?" "How can an 'arbitrary system of conventions' foretell the future?" "Why does common arithmetic serve the shopkeeper, the scientist and the mathematician all equally well?" Such questions form only a part of the general problem of the relation of language to reality. It is a commonplace that the scientist, like the theologian, creates the world in his own image, but what in fact is created is a language and neither the experiments of the one nor the postulates of the other show in what way language is tied to reality.

These essays presuppose some knowledge of, and familiarity with, the problems of the foundations of mathematics and do not undertake a historical survey of current theories. The applications of the formal system we construct are given in outline only and demand a not inconsiderable mathematical technique for their appreciation, but the system itself, both in form and content, is probably one of the simplest of its kind. The equation calculus was conceived some ten years ago and many of the thoughts herein expressed date from the same time. The first draft of the essays was completed a year later but the final revision and rewriting were delayed by the war; in the intervening years I have subtracted from, rather than added to, the original draft so that the essays may be said to be more sculptured than constructed.

Of the many friends who have helped, encouraged and inspired this work, first and foremost I must mention Ludwig Wittgenstein, to whose lectures at Cambridge between 1931–34 and the many conversations I was privileged to have with him, I am immensely indebted; only in recent years have I grown to understand how much he taught me.

To Paul Bernays I offer gratitude and appreciation for his support, advice and help given so generously for many years.

My last word is for my dear friend Francis Skinner, who died at Cambridge in 1941, and left no other record of his work and of his great gifts of heart and mind than lies in the recollections of those who had the good fortune to know him.

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August, 1949.

INTRODUCTION

EXISTENCE IN MATHEMATICS. ZENO'S PARODOX. THE INFINITUDE OF PRIMES.

The great discoveries in mathematics are not in the nature of uncovered secrets, pre-existing timeless truths, but are rather constructions: and that which is constructed is a symbolism, not a proposition. The power of a living symbolism is the source of that insight into mathematics which is termed mathematical intuition.

In the foundations of mathematics a *formal calculus* plays the part which is taken by symbolism in the informal development. A symbolism leads on, a formal calculus leads back, and just as a formal calculus, rightly is felt by creative mathematicians as a barrier to the free expression of ideas, so in the critical study of the foundations, symbolism is a source of error and misconception.

The foremost question of the foundations of mathematics for the last twenty-five years concerns the legitimacy of certain methods of proof in mathematics. What makes this question so difficult is the absence of any absolute standard, outside mathematics, with which mathematics can be compared. Philosophers have held that such a standard is to be found in a study of the Mind; that just as the laws of Nature are discovered by observation of, and experiment in, natural phenomena, so too the laws of mathematics are to be found as laws of thought, by a study of thinking processes. Yet, if we consider, we find that the 'Laws of Nature' are but empirical hypotheses, subject to limitation and modification, admitting exceptions, related to time and describing the world as it is, whereas the rules of mathematics *are* mathematics, timeless because they are outside time, independent of all observation and experiment and accordingly neither true nor false, expressing no property of the world, neither validating, nor validated by, any fact. The 'laws of thought', if by the term we mean laws formulated by experimental psychologists, no more form a standard by which the rules of mathematics can be tested, than the deductions of a Martian, from *observations* of the game, test the validity of the rules of chess.

What then is the meaning of the controversy between formalists and constructivists? The formalists say that the criteria by which formal systems are tested are the criteria of freedom-from-contradiction and completeness, and all their efforts in the past twenty-five years have been directed towards proving that a formal calculus, like *Principia*

Mathematica, a calculus of implication, disjunction and quantification, contains no insoluble problems, and in particular towards the construction of a proof of the non-contradictoriness of this calculus. This preoccupation with *contradiction* springs from two widely different sources. From the time when language first ceased to be only a *vehicle* of communication and became itself an *object* of discourse, men have invented paradoxes. Already in the oldest paradoxes of which we have written record, the paradox of the "Liar" and the infinity paradox of Zeno we find the prototypes of the paradoxes of the present day. The construction of formal systems, the very object of which was the resolution of these paradoxes, has accomplished only their multiplication. It seems as if the elimination of a paradox can, so to speak, be achieved on only one plane at a time and at the cost of fresh paradoxes on higher planes. Rather, this is the impression which the logistic technique of paradox resolution has produced, for in fact the roots of the paradoxes lie in this very technique.

The second source from which the fear of hidden, yet to be discovered, contradictions springs is the uncertainty which every thinker has felt, particularly in recent years, regarding the significance of postulational methods in mathematical philosophy, a feeling that the postulation of the existence of even a mathematical entity is entirely specious, metaphysical, and in no way comparable to the invention of a physical entity to serve as a medium of expression or a physical model.

Existence in Mathematics. Problems regarding the existence of mathematical entities are of many different kinds. Contrast the questions. Are there numbers, do numbers exist ? Does the real number "*e*" exist as something apart from the sequence $1,\ 1+1,\ 1+1+1/2\,!,\ 1+1+1/2\,!+1/3\,!, \ldots$? ; Is there a prime number greater than 10^{10} ? ; Is there a prime pair less than 10^{10} ? greater than 10^{10} ? To the first question one may answer : Amongst the *signs* of our language we distinguish the *numerals*, or number-signs, which are constructed from the number-sign "0" by the operation of placing a vertical stroke after a number sign ; the term 'number' is thus a classification index of signs. The sense in which we can say that numbers exist is that number signs are used in our language. Such questions as "have numbers an objective reality", "are numbers subjects or objects of thought" are disguised questions concerning the grammar of the word "number" and ask whether or not we formulate such sentences as : That which you see, hear, taste, touch, etc., are numbers.

The second question is concerned with the meaning of limit-processes in mathematics and with the concept of an infinite set. To say that the real number "*e*" has an existence independent of the convergent sequence $1,\ 1+1,\ 1+1+1/2\,!, \ldots$ is equivalent to saying that some infinite process is *completed*, for instance that the process of writing down *all* the digits in the decimal expansion of e has been carried through. In what sense can an infinite process be thought of as completed? An infinite process is, by definition, a process in which each stage of the process is followed by another stage just as each numeral is followed by another, formed by adding a vertical stroke to the end of the numeral. An infinite process is therefore an *unfinishable* process, a process which does not contain the possibility of being completed. A completed infinite process is a contradiction in terms.

The relation of Zeno's paradox to the formalist-finitist controversy.

It is, however, commonly argued that we *can* conceive of a completed infinite process; that in fact, were it not so, Zeno's famous argument would force us to deny the possibility of motion. For in passing from one position A, to another B, a body must pass through the mid-point A_1 of AB and then through the mid-point A_2 of A_1B, and then through the mid-point A_3 of A_2B, and so on. Thus the motion from A to B may be considered to consist in an unlimited (infinite) number of stages, viz., the stage of reaching A_1, the stage of reaching A_2, the stage of reaching A_3, and so on. After any stage A_n follows the stage A_{n+1} and no matter how many of the stages we have passed through we have not reached B, and so we *never* reach B. But if motion from a point A to any point B is not possible, then no motion is possible. Thus Zeno argues; and by *reductio ad absurdum* (for motion is certainly possible) it follows that the motion from A to B must be regarded as a completed infinite process. The fallacy in this discussion is by no means easy to detect and seems to have escaped the notice of many competent thinkers.

If we say that motion is possible we are appealing to our familiar experience of physical bodies changing their positions. Let us imagine a man running along a race track across which tapes are strung a few feet from the ground. We may suppose the track is 100 yards long and that we commence to string the tapes at the 50 yard mark. If we call the ends of the track A, B and the 50 yard mark A_1, then A_2 is the mid point of A_1B and so on as above. At each of the points A_1, A_2, A_3, . . . a tape is strung across the track. As a man runs from A to B he will break each of the tapes we set up, and if we suppose that a tape has been set up at each of the points A_1, A_2, A_3, . . . then the runner will have broken an infinite number of tapes. In putting the argument in this form we have only placed the difficulty in a more obvious light, for we are now confronted with the task of setting up an unlimited number of tapes, or, looking at it from another view point, of isolating an unlimited number of points. On the one hand we have the possibility of passing from A to B and the unlimited possibility of specifying points between A and B (an unlimited number of fractions between 0 and 100) and on the other hand the impossibility of isolating these points on the track. How is this apparent incompatibility resolved?

Think of a man *counting* from 0 to 100. He may say all the natural numbers from 0 to 100, or he may say only the "tens" or just "fifty", "hundred", or he may say "half, one, one and a half, two", and so on, by halves, up to a hundred. If he counts by tens can we say he has passed through all the integers between one and a hundred (or passed over them)? And if he counts by units, that he has passed through *all* the fractions between these units? One would not hesitate to answer that the man has counted, or passed through, only those numbers which in fact he counted, whatever they were, and that numbers which he did not count, even though such numbers could be inserted between the numbers which he counted, were *not* passed through by him in his counting. Correspondingly when a man runs from A to B he passes those points (or breaks those tapes) which we isolate, which we name, and these points only and what we name will be a finite number of points, however great. The Zeno argument achieves its end by confusing the physical possibility of motion with the logical possibility of naming as many points as we please.

It is sometimes maintained that the resolution of Zeno's paradox lies in the fact that a steadily increasing infinite sequence of numbers may be bounded; e.g., the sequence whose n^{th} term is $n/(n+1)$ is steadily increasing, because $(n+1)/(n+2)$ exceeds $n/(n+1)$ by $1/(n+1)(n+2)$, and is bounded above by unity since $n/(n+1)$ is $1/(n+1)$ less than 1. This fact is applied to Zeno's argument in the following way: Suppose the tape at the point A_1 is fixed in $\frac{1}{2}$ minute, the tape at the point A_2 in $(\frac{1}{2})^2$ minutes, the tape at the point A_3 in $(\frac{1}{2})^3$ minutes, and so on, then the first n tapes are fixed in $1-(\frac{1}{2})^n$ minutes, so that within one minute *all* the tapes are fixed, and an infinite operation has been completed. Thus although there always remains a tape to be fixed no matter how many have been set up, yet within a minute of starting there is no tape which has not yet been set up. This argument does not however resolve the paradox but merely restates it in a fresh plane, for the conclusion seems now to be that measurement of time is impossible, and this in its turn is bound up with the possibility of motion (for example, time may be measured by the motion of the hand of a clock, or the sun across the sky). The resolution of the paradox in this form is the same as the resolution of the motion-paradox. If our criterion for the number of tapes fixed in a minute is the criterion of experiment, then no matter how rapidly the experiment is carried out the *unfinishable* task of setting up an unlimited number of tapes, will not be finished. And if our criterion is just that $1-(\frac{1}{2})^n$ is less than unity, then this criterion tells us nothing about an actual experiment and we cannot appeal to the reality of the passage of time to generate the paradox.

Consider an analogous example. A line is drawn from the point 0 to the point 1. In what sense can we say that the line passes through infinitely many points, that *drawing* the line completes an infinity of operations, say the operations of joining 0, $\frac{1}{2}$ then $\frac{1}{2}$, $\frac{2}{3}$ then $\frac{2}{3}$, $\frac{3}{4}$ and so on? Let us describe two operations. (1) Drawing a line from the point 0 to the point 1, and (2) drawing a line from 0 to $\frac{1}{2}$, a line from $\frac{1}{2}$ to $\frac{2}{3}$, a line from $\frac{2}{3}$ to $\frac{3}{4}$, and so on. The first operation has but a single stage, the second is an unfinishable operation by definition, since no *last* stage is defined. What have these operations in common and in what way do they differ? Zeno would persuade us that the first operation is indistinguishable from the second, thereby generating the paradox of a finished operation being identical with an unfinishable one. In drawing a line from 0 to 1 we have certainly drawn a line from 0 to $\frac{1}{2}$, and a line from $\frac{1}{2}$ to $\frac{2}{3}$ and a line from $\frac{2}{3}$ to $\frac{3}{4}$, and so on, so that by carrying out the first operation, there is no stage of the second operation that is unfinished. The fallacy in this argument is concealed beneath a dual usage of the expression "a line is drawn from a point A to a point B". In describing the first operation, and in describing each of the stages of the second operation, the expression "a line drawn from a point A to a point B" means a line whose end-points are A and B, i.e., a physical mark, a stroke, terminating at A and B. The first operation consists in drawing a stroke from 0 to 1. The second operation consists in drawing successively strokes from 0 to $\frac{1}{2}$, from $\frac{1}{2}$ to $\frac{2}{3}$, from $\frac{2}{3}$ to $\frac{3}{4}$ and so on. Yet when we say that the stroke from 0 to 1 is also a stroke from 0 to $\frac{1}{2}$ (or $\frac{1}{2}$ to $\frac{2}{3}$, etc.) we have now changed the meaning of the expression "a stroke from A to B" for the stroke from 0 to 1 does *not* terminate at the point $\frac{1}{2}$ and the operation of drawing a stroke from 0 to 1 cannot be said to consist in strokes from 0 to $\frac{1}{2}$, from $\frac{1}{2}$ to $\frac{2}{3}$, etc. What constitutes a *stage* of the second operation, the termination of a stroke at one of the points $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, . . . is precisely what is lacking in the first operation.

The resolution of Zeno's paradox may be expressed by saying that Zeno confuses a literal and a metaphorical use of the expression "moving from one point to another". In the literal sense of this expression motion is change of the relative positions of physical objects, and 'point' is a physical object; in this sense motion from one point to another passes through but a finite number of 'points', physical objects isolated and specified on the route. We may specify as many such objects as we please, but what we specify will have a number. The metaphorical use of the expression "moving from one point to another" gives this expression the sense of "a variable increasing from one value to another". As the variable x increases from 0 to 1 it passes through the values $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, and so on, and therefore, seemingly an endless succession of events is completed. But the expression "as x increases from 0 to 1 it passes through the values $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, and so on", says only that the function $m/(m+1)$ is one which increases with m, all its values lying in $(0, 1)$. And the proof that the function is increasing and that its values lie in $(0, 1)$ does not involve the possibility of completing an endless process, for what is to be proved is just that $(m+1)/(m+2)$ exceeds $m/(m+1)$ by $1/(m+1)(m+2)$ and that unity exceeds $m/(m+1)$ by $1/(m+1)$, i.e., that $(m+1)^2 = m(m+2)+1$ and $(m+1)-1 = m$.

The infinitude of primes. We come now to the third question "Is there a prime number greater than 10^{10}" ? Consider first the question: Is there a prime number between 10^{10} and $10^{10} + 10$" ? The nine numbers $10^{10} + 1$, $10^{10} + 2$, $10^{10} + 3$, $10^{10} + 4$, $10^{10} + 5$, $10^{10} + 6$, $10^{10} + 7$, $10^{10} + 8$, $10^{10} + 9$, can be tested to find whether or not they are prime, that is to say, each of the numbers may be divided in turn by the numbers 2, 3, 4, 5, up to 10^5 and if one of the nine numbers leaves a remainder not less than unity for each of the divisions then that number is prime; if however each of the nine numbers leaves a zero remainder for some division then none of the nine numbers is prime. In the same way we can test whether any of the numbers between $10^{10} + 10$ and $10^{10} + 20$ is prime, and, of course, the test is applicable to any finite series (i.e., a series in which the *last* member is given). Thus the question "is there a prime number between a and b" may be decided one way or the other in a specifiable number of steps, depending only upon a and b, whatever numbers a and b may be. When, however, we ask whether there is a prime number greater than 10^{10} the test is no longer applicable since we have placed no bound on the number of experiments to be carried out. However many numbers greater than 10^{10} we tested, we might not find a prime number and yet should remain *always* unable to say that there was no prime greater than 10^{10}. We might, in the course of the experiment, chance upon a prime number, but unless this happened the test would be inconclusive. To show that the test can really be decisive it is necessary that we should be able in some way to limit the number of experiments required, and this was achieved by Euclid when he proved that, for each value of n, the chain of numbers from n to $n! + 1$ inclusive, contains at least one prime number. The underlying ideas of this proof are just that $n! + 1$ leaves the remainder unity when divided by any of the numbers from 2 to n, and that the *least* number, above unity, which divides any number is necessarily prime (every number has a divisor greater than unity, namely, the number itself, and the least divisor is prime since its factors will also divide the number and so must be unity or the least divisor itself); thus the *least* divisor (greater than unity) of $n! + 1$ is prime and greater than n. What Euclid's proof accomplished is not the discovery or specification of a prime number but the construction of a function whose values are prime numbers. We shall have further occasion to observe how often mathematics answers the question "is there a number with such and such properties" by *constructing* a function; the manner and kind of such constructions will form the subject of later considerations.

When we turn to the question concerning the existence of a prime pair greater than 10^{10} we are faced with the *endless* task of testing, one after the other, the primes greater than 10^{10}, of which, as we have seen, we can determine as many as we please, to find whether there are two primes which differ by 2. In this instance no function has been constructed whose values form prime pairs, and there is no way of deciding the question negatively. We have asked a question—if *question* it be

—to which there is no possibility of answering *no* and to which the answer *yes* could be given only if we *chanced* to find, in the course of the endless task of seeking through a succession of primes, a pair of primes which differed by 2. The formalists maintain that we can conceive of this endless task as completed and accordingly can say that the sentence "there is a prime pair greater than 10^{10}" must be either true or false; to this, constructivists reply that a "completed endless task" is a self-contradictory concept and that the sentence "there is a prime pair greater than 10^{10}" may be true but could never be shown to be false, so that if it be a defining characteristic of sentences that they be either true or false (the principle of the excluded middle) then "there is a prime pair greater than 10^{10}" is no sentence. This dilemma has led some constructivists to deny the principle of the excluded middle, which means they have changed the definition of "sentence", others to retain the principle, and, albeit unwillingly, reject the unlimited existential proposition, whilst the formalist retains both the principle of excluded middle and the unlimited existential proposition together with an uneasy preoccupation with the problem of freedom-from-contradiction. The real dispute between formalists and constructivists is not a dispute concerning the legitimacy of certain methods of proof in mathematics; the constructivists deny and the formalists affirm the possibility of completing an endless process.