Function Spaces, Differential Operators and Nonlinear Analysis

The Hans Triebel Anniversary Volume

$$\| f \,|\, F^s_{p,q}(\mathbb{R}^n) \| = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \left| \left(\phi_j \widehat{f} \right)^{\vee}(\cdot) \right|^q \right)^{1/q} \Big| L_p(\mathbb{R}^n) \right\|$$

Dorothee Haroske
Thomas Runst
Hans-Jürgen Schmeisser
Editors

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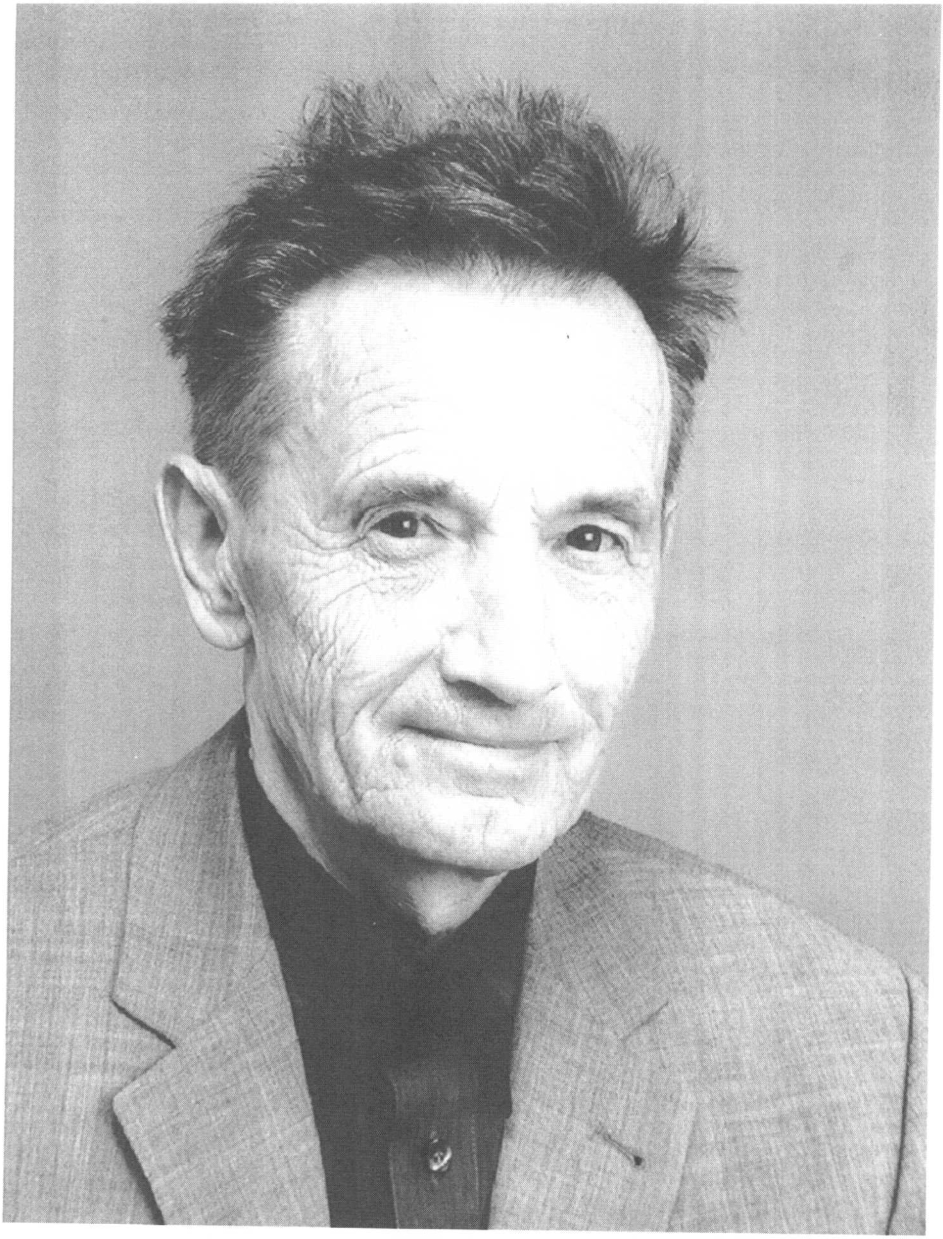

This collection of research papers is dedicated to
Hans Triebel
on the occasion of his 65th birthday

Preface

This volume is dedicated to our teacher and friend Hans Triebel. The core of the book is based on lectures given at the International Conference "Function Spaces, Differential Operators and Nonlinear Analysis" (FSDONA–01) held in Teistungen, Thuringia / Germany, from June 28 to July 4, 2001, in honour of his 65th birthday. This was the fifth in a series of meetings organised under the same name by scientists from Finland (*Helsinki, Oulu*), the Czech Republic (*Prague, Plzeň*) and Germany (*Jena*) promoting the collaboration of specialists in East and West, working in these fields.

This conference was a very special event because it celebrated Hans Triebel's extraordinary impact on mathematical analysis. The development of the modern theory of function spaces in the last 30 years and its application to various branches in both pure and applied mathematics is deeply influenced by his lasting contributions. In a series of books Hans Triebel has given systematic treatments of the theory of function spaces from different points of view, thus revealing its interdependence with interpolation theory, harmonic analysis, partial differential equations, nonlinear operators, entropy, spectral theory and, most recently, analysis on fractals.

The presented collection of papers is a tribute to Hans Triebel's distinguished work. The book is subdivided into three parts:

- Part I contains the two invited lectures by O.V. Besov (*Moscow*) and D.E. Edmunds (*Sussex*) having a survey character and honouring Hans Triebel's contributions.
- The papers in Part II reflect seven recent developments in the theory of function spaces, linear and nonlinear partial differential equations presented by outstanding experts in the field.
- Shorter communications related to the topics of the conference and Hans Triebel's research are collected in Part III.

Hans Triebel's personal qualities leave a lasting impression on his colleagues, students and friends. Many of us have benefited from his extensive knowledge, his ideas and hospitality. We are glad to have the opportunity to express our deep gratitude to him.

We acknowledge with gratitude financial support by the DFG, the Graduiertenkolleg *"Analytic and stochastic structures and systems"* and the Friedrich-Schiller-University Jena. It is a pleasure for us to give our special thanks to our colleagues Michele Bricchi, Serguei Dachkovski, Hans-Gerd Leopold and Winfried

Sickel for many helpful suggestions concerning the present book and the great amount of work they did to make things run smoothly. We are very much indebted to Birkhäuser Verlag for their important support during the conference and for this project. Finally, we would like to thank all the speakers and all participants in the conference FSDONA-01 for their energy and enthusiasm which made its success possible.

Jena, in April 2002

Dorothee D. Haroske
Thomas Runst
Hans-Jürgen Schmeißer

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Part I

D. Haroske, T. Runst, H.-J. Schmeisser (eds.): Function Spaces, Differential Operators and Nonlinear Analysis. The Hans Triebel Anniversary Volume.

Spaces of Differentiable Functions

Oleg Besov[1] and Gennadiy Kalyabin[2]

Dedicated to Prof. Hans Triebel on the occasion of his 65th birthday

Introduction

A brilliant exposition of numerous aspects of the theory of function spaces (embeddings and equivalent norms, description in terms of smoothness properties; decompositions and approximations; interpolation via real and complex methods; trace problems; extension operators for regular and irregular domains; applications to *PDO* and ΨDO etc.) is given in the series of famous books [T-78], [T-83], [T-92], [T-01] by Professor Hans Triebel. Some other approaches one may find in fundamental monographs [S-88], [N-75], [St-70], [M-85], [dVLo-93], [BIN-96] and survey papers [KL-87], [BKuLN-90] where the detailed references are given.

The goal of this paper is to supplement these texts (avoiding the reproduction of passages from them) with some selected topics which touch mainly the historically significant points and with more recent results obtained by mathematicians of Russia in the theory of function spaces.

In Section 1 a brief history is expounded of the most known types of function spaces: Sobolev spaces W_p^l, Nikol'skiy spaces H_p^s, Lipschitz type spaces B_{pq}^s and the Lizorkin-Triebel spaces L_{pq}^s (often referred to also as Triebel-Lizorkin spaces F_{pq}^s). The spaces of functions of generalized smoothness are surveyed in Section 2 where the results on capacities are of major interest. The embedding theory for spaces on domains (Section 3) contains many quite fresh results related to regular and irregular domains. Function spaces on domains which allow the extension onto the whole Euclidean space R^n and corresponding extension operators are described in Section 4. In the final Section 5 diverse additional questions are briefly touched upon.

The definitions, theorems, remarks are given not as separate parts but are simply set forth in the text with necessary references.

Authors would like to express their deep gratitude to A. Kufner, V. Maz'ya, S. Pohozaev for valuable remarks which are taken into account.

[1] This work was supported by grants of RFBR-99-01-00868, LSSRF-00-15-96047 and INTAS-99-01080

[2] This work was supported by the grant of RFBR-99-01-00868

It is important to emphasize that the active cooperation between Russian and German mathematicians (more widely: between scientists of the former Soviet Union – fSU – and those of Western countries) initiated by Professor Hans Triebel and the Mathematical Institute of Friedrich-Schiller-University (FSU), Jena, has played a significant and helpful role in the achievements in function spaces theory during the last decades.

1. Brief history

In [T-83: 2.2.1] three periods in function spaces theory are marked out: the classical period (Lebesgue spaces L_p, the spaces C, C^m, Hölder spaces $C^s, s > 0$, Hardy classes H_p of analytic functions); the constructive period (1930–75) and the period of systematization which is continuing until now. In the sequel we deal with the two last periods.

1.1. Sobolev spaces

The theory of spaces of differentiable functions of several real variables originates from the paper by S.L. Sobolev [S-38]. He proposed the notion of generalized derivative and introduced the Banach spaces $W_p^l(\Omega)$ of functions f, defined on a domain $\Omega \subset R^n$ with the norm

$$\| f \mid W_p^l(\Omega)\| := \sum_{|\alpha|\le l} \| D^\alpha f \mid L_p(\Omega)\|, \tag{1.1}$$

where $l \in \mathbf{N}$, $1 \le p < \infty$, $L_p(\Omega)$ is the Lebesgue space of functions with the norm

$$\| f \mid L_p(\Omega)\| := \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p}. \tag{1.2}$$

These Sobolev spaces and their generalizations play until now an important role in the theory of function spaces and in applications to mathematical physics. It was Sobolev who first obtained the embedding theorems which are related to a domain satisfying the cone condition. For a function $f \in W_p^l(\Omega)$ they assert its summability to the power $q > p$ on the same domain $\Omega = \Omega^{(n)} \subset R^n$ or on sufficiently smooth manifolds $\Omega^{(m)}$ of the dimension $m < n$, belonging to Ω:

$$W_p^l(\Omega) \subset L_q(\Omega^{(m)}) \quad \text{for } \delta := l - \frac{n}{p} + \frac{m}{q} > 0, \tag{1.3}$$

where $1 \le p < q \le \infty$, $1 \le m \le n$; in the case $m = n$, $q = \infty$ one may replace $L_\infty(\Omega)$ in (1.3) by the space $C(\Omega)$ of all functions continuous and bounded on Ω.

For $q < \infty$ the embedding (1.3) holds also if $\delta \ge 0$. These supplements are contained in results by S.L. Sobolev ($p > 1$, $m = n$), V.P. Il'in [I-54] ($p > 1$), E. Gagliardo ($p = 1$) (cf. [BIN-96: Ch. 3]). V.I. Kondrashov (1945) has established that the embedding (1.3) is *compact* provided the domain is bounded and $\delta > 0$. The first method used by Sobolev to prove the embeddings was the integral representation method in which the function is decomposed into certain sums of potential type integrals applied to the function itself and to its derivatives.

The critical case $m = n = lp$ was studied by S.I. Yudovich (1961) who proved that for $1 < p < \infty$ the space $W_p^l(R^n)$ is embedded into Orlicz space $L_M^\star$ with M being the Young function $M(u) := \exp(u^{p'})$ (see [BIN-96: 10.6] for more historical comments).

1.2. Nikol'skiy spaces

The next important step was made in the 1950-ies by S.M. Nikol'skiy who introduced the function spaces $H_p^s(R^n)^1$, $1 \le p \le \infty$, $s > 0$, and established the theory of these spaces. A function $f \in H_p^s(R^n)$ if f is defined on the Euclidean space R^n and has the finite norm

$$\| f \mid H_p^s(R^n)\| := \| f \mid L_p(R^n)\| + \sum_{j=1}^{n} \sup_{h>0} h^{-s+k} \| \Delta_{j,h}^M D_j^k f \mid L_p(R^n)\|, \quad (1.4)$$

where $M > 0, k \ge 0$ are integers such that $k < s$, $M + k > s$, $D_j^k f(x) := \partial^k f / \partial x_j^k$ are the generalized derivatives and

$$\Delta_{j,h}^M f(x) := \sum_{l=0}^{M} (-1)^{(M+l)} C_M^l \, f(x + lhe_i) \quad (1.5)$$

are the coordinate-wise differences with the step $h > 0$ of the M-th order.

For different admissible pairs (M, k) the norms (1.4) are equivalent to each other and to the following *equivalent norming in terms of approximation via entire functions of exponential type* [N-51]:

$$\| f \mid H_p^s(R^n)\|^{(A)} := \| f \mid L_p(R^n)\| + \sup_{N>1} N^s \inf_{g_N \in \mathrm{EFET}(N)} \|f - g_N \mid L_p(R^n)\|, \quad (1.6)$$

Here EFET(N) stands for a set of all entire functions of the exponential type N which in 2π-periodical case coincides with the set of all trigonometric polynomials of the degree $\le N$ (cf. [N-75: Ch. 3]). Of great significance are the Bernstein-Nikol'skiy inequalities for the class EFET(N)

$$\| D^\alpha g_N \mid L_q(R^m)\| \le c_{p,q} N^{|\alpha| + \frac{n}{p} - \frac{m}{q}} \| g_N \mid L_p(R^n)\|, \quad 1 \le p \le q \le \infty. \quad (1.7)$$

Having based on these relationships S.M. Nikol'skiy obtained the embedding theorem for H_p^s-spaces:

$$H_p^s(R^n) \subset H_q^\rho(R^m), \; 1 \le p \le q \le \infty, \; 1 \le m \le n, \; \rho = s - \frac{n}{p} + \frac{m}{q} > 0 \quad (1.8)$$

which establishes for $m < n$ the integral-differential properties of the *trace* of the function $f \in H_p^s(R^n)$ on the m-dimensional subspace R^m. It is important that for $q = p$ this theorem is reversible, i.e. any function $\varphi \in H_p^\rho(R^m)$ is a trace of some function $f \in H_p^s(R^n)$, thus the sharp description of the trace space is given.

[1] Here H_p^s does not denote the Bessel-potential (or Lebesgue or Liouville) spaces (defined in Section 1.4).

1.3. Lipschitz spaces of functions

The statement of the complete results on the trace problem of Sobolev spaces required the consideration of Lipschitz spaces of functions. The first results were obtained by N. Aronszajn for $p = 2$ [A-55]. L.N. Slobodeckiy built the theory of the anisotropic Sobolev spaces $W_2^l(R^n)$, $l = (l_1, \dots, l_n)$ with integer and fractional smoothness parameters [Sl-58]. E. Gagliardo characterized the traces of functions of Sobolev space $W_p^1(R^n)$ on the hyperplanes in R^n [Ga-57].

In the 1960-ies O.V. Besov has studied the 4-parametric family scale of function spaces $B_{p,q}^s(R^n)$, $s > 0$, $1 \le p$, $q \le \infty$, $n \in \mathbf{N}$ defined by the finiteness of the norm

$$\| f \mid B_{p,q}^s(R^n)\| := \| f \mid L_p(R^n)\| + \sum_{j=1}^{n} \left\{ \int_0^1 \left(\frac{\| \Delta_{j,h}^M D_j^k f \mid L_p(R^n)\|}{h^{s-k}} \right)^q \frac{dh}{h} \right\}^{1/q} . \qquad (1.9)$$

It is clear that $B_{p,\infty}^s(R^n) = H_p^s(R^n)$. The embeddings analogous to (1.8) are valid for B-spaces as well as the description of traces on R^m. The spaces $B_{p,q}^s(R^n)$ are closely related to Sobolev spaces: for s integer and $p = q = 2$ one has $B_{2,2}^s(R^n) = W_2^s(R^n)$, and for $1 < p < \infty$ in terms of the B-spaces the traces on R^m for Sobolev spaces are described:

$$W_p^s(R^n)\Big|_{R^m} = B_{p,p}^{s-\frac{n-m}{p}}(R^m) \quad \text{for} \quad s - \frac{n-m}{p} > 0 . \qquad (1.10)$$

The latter result for $m = n - 1$, $s = 1$, $p = 2$ was established by N. Aronszajn and for $1 < p < \infty$ by E. Gagliardo. The general case $1 \le m < n - 1$ was studied by O.V. Besov [B-61] (see details in [N-75: Introduction]).

1.4. Lizorkin-Triebel spaces

P.I. Lizorkin [L-72] has introduced the spaces $L_{p,q}^s(R^n)$ $(1 < p, q < \infty, s > 0)$ with the norms

$$\| f \mid L_{p,q}^s(R^n)\| := \| \, \|\{2^{ks} f_k(x)\}|l_q\| \mid L_p(R^n) \, \| \qquad (1.11)$$

where $f = \sum_{k=1}^{\infty} f_k$ is the decomposition of the function f:

$$f_1(x) := S_1 f(x), \quad f_k(x) := S_{2^k} f(x) - S_{2^{k-1}} f(x), \; k \ge 1 \qquad (1.12)$$

and S_N in Fourier images correspond to the multiplication by the characteristic function of the cube $Q_N := \{\xi \in R^n : \max |\xi_j| \le N\}$. The paper [L-67] in which the $L_p(R^n, l_q)$-norms first appear in function spaces theory also should be mentioned.

A little bit later H. Triebel (cf. [T-83: 2.3.5]) suggested to use instead of the cutting operators S_N the smooth decompositions of the Fourier transform

$$\varphi_1(\xi) := \varphi(\xi) \in C_0^\infty(R^n); \quad \varphi(\xi) = 1, \; |\xi| \le 1; \quad \varphi(\xi) = 0, \; |\xi| \ge 2$$

$$\varphi_k(\xi) := \varphi(2^{-k}\xi) - \varphi(2^{-k+1}\xi), \; k > 1 \qquad (1.13)$$

so that $\sum_{k=1}^{\infty} \varphi_k(\xi) \equiv 1$.

For $0 < p < \infty$, $0 < q \le \infty$, $-\infty < s < \infty$ the space $F^s_{p,q}(R^n)$ consists by definition of all functions f which belong to the Schwartz space of distributions $S'(R^n)$ and possess the finite (quasi)-norms:

$$\| f \mid F^s_{p,q}(R^n)\|^{(\varphi)} := \| \ \|\{2^{ks}F^{-1} \ \varphi_k(\xi)Ff(x)\}| \ l_q\| \mid L_p(R^n)\| \qquad (1.14)$$

where $F,\ F^{-1}$ are the direct and the inverse Fourier transforms (for different $\varphi(\xi)$, satisfying (1.13) one obtains equivalent norms). It is important that for the range of parameters considered by Lizorkin (i.e. for $1 < p, q < \infty,\ s > 0$) these two definitions yield the same space: $L^s_{p,q}(R^n) = F^s_{p,q}(R^n)$. With this case we shall deal in the sequel.

For $q = 2$ Lizorkin-Triebel spaces coincide with Liouville spaces $L^s_p(R^n)$ (= Bessel potentials spaces) [T-83: 2.2.1]:

$$\| f \mid L^s_p(R^n)\| := \|F^{-1}(\ (1+|\xi|^2)^{s/2} \ Ff(\xi) \ )(x)| \ L_p(R^n)\|. \qquad (1.15)$$

Note that the norm in the space $B^s_{p,q}(R^n)$ differs from the norm in $L^s_{p,q}(R^n)$ only by interchanging the order of taking $L_p(R^n)$- and l_q-norms in (1.13). One of Triebel's observations (sometimes called "the thumb rule") is that almost all propositions established for L-spaces are valid (and can be proved much easier) for B-spaces as well. So further in this paper we shall concentrate our attention mostly on Lizorkin-Triebel spaces.

In [T-83: 2.5], [T-92: Chs 2,4] one may find many results related to the equivalent norms in terms of differences, local averaged oscillations and other smoothness characteristics, to the trace problem (on hyperplanes R^m and Lipschitz domains) for the whole scale $F^s_{p,q}(R^n)$. For the spaces $L^s_{p,q}(R^n), 1 < p, q < \infty,\ s > 0$ these results have been first established by Kalyabin (1977, 79, 81, 83). It turned out (rather unexpectedly) that the trace spaces on R^m, the criteria of embedding into the space $C(R^n)$, the multiplicativity property of Lizorkin-Triebel spaces are *not at all influenced by the second summability parameter* $q \in (1, \infty)$ in full contrast to B-spaces (for more details see further Section 2 and [KL-87]).

1.5. Other types of function spaces

The study of anisotropic spaces $H^{\overline{s}}_p$, $\overline{s} := (s_1, s_2, \dots s_n)$ with various differential properties in various coordinate directions has been initiated by S.M. Nikol'skiy. Later he also proposed another type of anisotropy, namely the mixed $L_{\overline{p}}$-norms, $\overline{p} := (p_1, \dots, p_n)$

$$\| f \mid L_{\overline{p}}(R^n)\| := \left(\int \left(\cdots \left(\int |f(x_1, \dots, x_n)|^{p_1} dx_1\right)^{\frac{p_2}{p_1}} \cdots\right)^{\frac{p_n}{p_{n-1}}} dx_n\right)^{\frac{1}{p_n}} \quad (1.16)$$

which appeared to be of great importance (detailed exposition and precise references may be found in [BIN-96: Ch. 3]).

S.M. Nikol'skiy was also the first who investigated the spaces $SH^{\overline{s}}_p$ defined in terms of mixed differences properties. This approach was further developed by T. Amanov (1965) (the spaces $SB^{\overline{s}}_{p,q}$). H.-J. Schmeisser (1980) has constructed a similar theory for the more complicated case of F-type spaces. The idea of these spaces may be well understood from their important particular case of the so-called Liouville-type spaces. By definition the space $SL_p^{\overline{s}}$, $\overline{s} := (s_1, s_2, \dots s_n)$, $s_j > 0$ is the set of all functions $f \in L_p(R^n)$ with the finite norm

$$\| f|SL_p^{\overline{s}} \| := \|F^{-1}(1 + \prod_{j=1}^{n} |\xi_j|^{2s_j})^{1/2} \, Ff| \, L_p(R^n) \| \,. \tag{1.17}$$

For all integer s_j the norm in $SL_p^{\overline{s}}$ is equivalent to the sum of L_p-norms of f and its mixed derivative $D_{x_1}^{s_1} D_{x_2}^{s_2} \dots D_{x_n}^{s_n} f$. The complete system of embeddings and equivalent norms for the scales $L_p^{\overline{s}}$, $SB_{p,q}^{\overline{s}}$, $SF_{p,q}^{\overline{s}}$ (known as the *spaces with dominated mixed derivatives*) one can find in the book [SchT-87].

2. Spaces of generalized smoothness

Many problems in analysis demand tools which would allow to measure the smoothness not by a single number (or several numbers) but by a comparison with certain calibre functions or calibre sequences. The idea of such spaces is ascending to H. Weyl. P.L. Ul'yanov (1967) has considered the spaces H_p^{ω} in the one-dimensional case. More general B-type spaces have been studied by M.L. Gol'dman (1972) and G.A. Kalyabin (1976). All details may be found in [KL-87] and [BKuLN-90] where the complete system of embeddings, equivalent normings in various terms, criteria of multiplicativity and some applications of spaces of generalized smoothness are expounded. Below only a small portion of these results are presented.

2.1. Generalized spaces of L- and B-type

Let us introduce two parameters-sequences of positive numbers $\{\alpha_k\}$ and $\{N_k\}$ such that for some $c_0 > 1$, $c_1 > 1$ and all k

$$\alpha_{k+1} \leq c_1 \alpha_k; \; \{\alpha_k^{-1}\} \in l_{q'} \text{ or resp. } l_{p'}; \quad N_{k+1} \geq c_0 N_k \tag{2.1}$$

and define the space $B_{p,q}^{(\alpha,N)}(R^n)$ (or resp. $L_{p,q}^{(\alpha,N)}(R^n)$) as the set of all functions $f \in L_p(R^n)$ for which there exist sequences $\{f_k(x)\} \in \text{EFET}(N_k)$ such that

$$\| \, \|\{\alpha_k \|f_k - f|L_p\|\}|l_q\| < \infty, \text{ resp.} \| \, \|\{\alpha_k(f_k(x) - f(x))\}|l_q\| \, |L_p\| < \infty. \tag{2.2}$$

The norm in the corresponding space is equal to the infimum of the quantity (2.2) over all sequences $\{f_k(x)\} \in \text{EFET}(N_k)$.

Restrictions (2.1) imposed on $\{\alpha_k\}$, $\{N_k\}$ are motivated by two requirements:

- (i) the space $B(L)_{p,q}^{(\alpha,N)}(R^n)$ does not coincide with the whole L_p,
- (ii) the operators of dilation: $f(x) \to f(tx)$, $t > 0$ are bounded in the corresponding space.

Thus the spaces of analytic functions (or Gevrey type spaces) are not embraced by these scales. Recall that the ordinary (power-scaled) spaces $L_{p,q}^s, B_{p,q}^s$ correspond to the case $\alpha_k := 2^k$, $N_k := 2^{k/s}$.