The Finite Element Method for Elliptic Problems

Philippe G. Ciarlet

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Philippe G. Ciarlet

Université Pierre et Marie Curie
Paris, France

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Society for Industrial and Applied Mathematics
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PREFACE TO THE CLASSICS EDITION

Although almost 25 years have elapsed since the manuscript of this book was completed, it is somewhat comforting to see that the content of Chapters 1 to 6, which together could be summarized under the title "The Basic Error Estimates for Elliptic Problems," is still essentially up-to-date. More specifically, the topics covered in these chapters are the following:

- description and mathematical analysis of various problems found in linearized elasticity, such as the membrane and plate equations, the equations of three-dimensional elasticity, and the obstacle problem;
- description of conforming finite elements used for approximating second-order and fourth-order problems, including composite and singular elements;
- derivation of the fundamental error estimates, including those in maximum norm, for conforming finite element methods applied to second-order problems;
- derivation of error estimates for the obstacle problem;
- description of finite element methods with numerical integration for second-order problems and derivation of the corresponding error estimates;
- description of nonconforming finite element methods for second-order and fourth-order problems and derivation of the corresponding error estimates;
- description of the combined use of isoparametric finite elements and isoparametric numerical integration for second-order problems posed over domains with curved boundaries and derivation of the corresponding error estimates;
- derivation of the error estimates for polynomial, composite, and singular finite elements used for solving fourth-order problems.

Otherwise, the topics considered in Chapters 7 and 8 have since undergone considerable progress. Additionally, new topics have emerged that often address the essential issue of the actual implementation of the finite element method. The interested reader may thus wish to consult the following more recent books, the list of which is by no means intended to be exhaustive:

- for further types of error estimates, a posteriori error estimates, locking phenomena, and numerical implementation: Brenner and Scott (1994), Wahlbin (1991, 1995), Lucquin and Pironneau (1998), Apel (1999), Ainsworth and Oden (2000), Bramble and Zhang (2000), Frey and George (2000), Zienkiewicz and Taylor (2000), Babuska and Strouboulis (2001), Braess (2001);
- for mixed and hybrid finite element methods: Girault and Raviart (1986), Brezzi and Fortin (1991), Robert and Thomas (1991);
- for finite element approximations of eigenvalue problems: Babuska and Osborn (1991);
- for finite element approximations of variational inequalities: Glowinski (1984);
- for finite element approximations of shell problems: Bernadou (1995), Bathe (1996);
- for finite element approximations of time-dependent problems: Raviart and Thomas (1983), Thomée (1984), Hughes (1987), Fujita and Suzuki (1991).

Last but not least, it is my pleasure to express my sincere thanks to Sara J. Triller, Arjen Sevenster, and Gilbert Strang, whose friendly cooperation made this reprinting possible.

Philippe G. Ciarlet
October 2001

BIBLIOGRAPHY

AINSWORTH, M.; ODEN, J.T. (2000): *A Posteriori Error Estimation in Finite Element Analysis*, John Wiley, New York.

APEL, T. (1999): *Anisotropic Finite Elements: Local Estimates and Applications*, Teubner, Leipzig.

BABUSKA, I.; OSBORN, J. (1991): Eigenvalue problems, in *Handbook of Numerical Analysis, Volume* II (P.G. Ciarlet & J.L. Lions, Editors), pp. 641–787, North-Holland, Amsterdam.

BABUSKA, I.; STROUBOULIS, T. (2001): *The Finite Element Method and Its Reliability*, Oxford University Press.

BATHE, K.J. (1996): *Finite Element Procedures*, Prentice-Hall, Englewood Cliffs, NJ.

BERNADOU, M. (1995): *Finite Element Methods for Thin Shell Problems*, John Wiley, New York.

BRAMBLE, J.H.; ZHANG, X. (2000): The analysis of multigrid methods, in *Handbook of Numerical Analysis, Volume* VII (P.G. Ciarlet & J.L. Lions, Editors), pp. 173–415, North-Holland, Amsterdam.

BRAESS, D. (2001): *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*, Second Edition, Cambridge University Press.

BRENNER, S.C.; SCOTT, L.R. (1994): *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, Berlin.

BREZZI, F.; FORTIN, M. (1991): *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, Berlin.

FREY, P.J.; GEORGE, P.L. (2000): *Mesh Generation: Application to Finite Elements*, Hermes Science Publishing, Oxford.

FUJITA, H.; SUZUKI, T. (1991): Evolution problems, in *Handbook of Numerical Analysis, Volume* II (P.G. Ciarlet & J.L. Lions, Editors), pp. 789–928, North-Holland, Amsterdam.

GIRAULT, V.; RAVIART, P.A. (1986): *Finite Element Methods for Navier–Stokes Equations*, Springer-Verlag, Berlin.

GLOWINSKI, R. (1984): *Numerical Methods for Nonlinear Variational Problems*, Springer-Verlag, Berlin.

HUGHES, T.J.R. (1987): *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*, Prentice-Hall, Englewood Cliffs, NJ.

LUCQUIN, B.; PIRONNEAU, O. (1998): *Introduction to Scientific Computing*, John Wiley, New York.

RAVIART, P.A.; THOMAS, J.M. (1983): *Introduction à l'Analyse Numérique des Equations aux Dérivées Partielles*, Masson, Paris (since 1998: Dunod, Paris).

ROBERT, J.E.; THOMAS, J.M. (1991): Mixed and hybrid methods, in *Handbook of Numerical Analysis, Volume* II (P.G. Ciarlet and J.L. Lions, Editors), pp. 523–639, North-Holland, Amsterdam.

THOMÉE, V. (1984): *Galerkin Finite Element Methods for Parabolic Problems*, Lecture Notes in Mathematics, Vol. 1054, Springer-Verlag, Berlin.

WAHLBIN, L.B. (1991): Local behavior in finite element methods, in *Handbook of Numerical Analysis, Volume* II (P.G. Ciarlet & J.L. Lions, Editors), pp. 353–522, North-Holland, Amsterdam.

WAHLBIN, L.B. (1995): *Superconvergence in Galerkin Finite Element Methods*, Lecture Notes in Mathematics, Vol. 1605, Springer-Verlag, Berlin.

ZIENKIEWICZ, O.C.; TAYLOR, R.L. (2000): *The Finite Element Method, Volume* I: *The Basis*, 5th edition, John Wiley, New York.

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CHAPTER 1

ELLIPTIC BOUNDARY VALUE PROBLEMS

Introduction

Many problems in elasticity are mathematically represented by the following minimization problem: The unknown u, which is the *displacement* of a *mechanical system*, satisfies

$$u \in U \quad \text{and} \quad J(u) = \inf_{v \in U} J(v),$$

where the set U of *admissible displacements* is a closed convex subset of a Hilbert space V, and the *energy* J of the system takes the form

$$J(v) = \tfrac{1}{2}a(v, v) - f(v),$$

where $a(\cdot, \cdot)$ is a symmetric bilinear form and f is a linear form, both defined and continuous over the space V. In Section 1.1, we first prove a general existence result (Theorem 1.1.1), the main assumptions being the *completeness* of the space V and the *V-ellipticity* of the bilinear form. We also describe other formulations of the same problem (Theorem 1.1.2), known as its *variational formulations*, which, in the absence of the assumption of symmetry for the bilinear form, make up *variational problems* on their own. For such problems, we give an existence theorem when $U = V$ (Theorem 1.1.3), which is the well-known *Lax–Milgram lemma.*

All these problems are called *abstract problems* inasmuch as they represent an "abstract" formulation which is common to many examples, such as those which are examined in Section 1.2.

From the analysis made in Section 1.1, a candidate for the space V must have the following properties: It must be complete on the one hand, and it must be such that the expression $J(v)$ is well-defined for all functions $v \in V$ on the other hand (V is a "space of finite energy"). The *Sobolev spaces* fulfill these requirements. After briefly mentioning some of their properties (other properties will be introduced in later sections, as needed), we examine in Section 1.2 specific examples of the abstract problems of Section 1.1, such as the *membrane problem*, the *clamped plate problem*, and the *system of equations of linear elasticity*, which is by far the most significant example. Indeed, even though throughout this book we will often find it convenient to work with the simpler looking problems described at the beginning of Section 1.2, it must not be forgotten that these are essentially convenient *model problems* for the system of linear elasticity.

Using various *Green's formulas* in Sobolev spaces, we show that when solving these problems, one solves, at least *formally*, elliptic boundary value problems of the second and fourth order posed in the classical way.

1.1. Abstract problems

The symmetric case. Variational inequalities

All functions and vector spaces considered in this book are real.

Let there be given a normed vector space V with norm $\|\cdot\|$, a *continuous* bilinear form $a(\cdot,\cdot): V\times V\to \mathbf{R}$, a *continuous* linear form $f: V\to\mathbf{R}$ and a non empty subset U of the space V. With these data we associate an *abstract minimization problem*: Find an element u such that

$$u\in U \quad \text{and} \quad J(u)=\inf_{v\in U} J(v), \tag{1.1.1}$$

where the functional $J: V\to\mathbf{R}$ is defined by

$$J: v\in V\to J(v)=\tfrac{1}{2}a(v,v)-f(v). \tag{1.1.2}$$

As regards existence and uniqueness properties of the solution of this problem, the following result is essential.

Theorem 1.1.1. *Assume in addition that*

(i) *the space V is complete,*
(ii) *U is a closed convex subset of V,*
(iii) *the bilinear form $a(\cdot,\cdot)$ is symmetric and V-elliptic, in the sense that*

$$\exists\alpha>0,\quad \forall v\in V,\quad \alpha\|v\|^2\leq a(v,v). \tag{1.1.3}$$

Then the abstract minimization problem (1.1.1) *has one and only one solution.*

Proof. The bilinear form $a(\cdot,\cdot)$ is an inner product over the space V, and the associated norm is equivalent to the given norm $\|\cdot\|$. Thus the space V is a Hilbert space when it is equipped with this inner product. By the Riesz representation theorem, there exists an element $\sigma f \in V$ such that

$$\forall v \in V, \quad f(v) = a(\sigma f, v),$$

so that, taking into account the symmetry of the bilinear form, we may rewrite the functional as

$$J(v) = \tfrac{1}{2}a(v, v) - a(\sigma f, v) = \tfrac{1}{2}a(v - \sigma f, v - \sigma f) - \tfrac{1}{2}a(\sigma f, \sigma f).$$

Hence solving the abstract minimization problem amounts to minimizing the distance between the element σf and the set U, with respect to the norm $\sqrt{a(\cdot,\cdot)}$. Consequently, the solution is simply the projection of the element σf onto the set U, with respect to the inner product $a(\cdot,\cdot)$. By the projection theorem, such a projection exists and is unique, since U is a closed convex subset of the space V. □

Next, we give equivalent formulations of this problem.

Theorem 1.1.2. *An element u is the solution of the abstract minimization problem* (1.1.1) *if and only if it satisfies the relations*

$$u \in U \quad \textit{and} \quad \forall v \in U, \quad a(u, v - u) \geq f(v - u), \tag{1.1.4}$$

in the general case, or

$$u \in U \quad \textit{and} \quad \begin{cases} \forall v \in U, & a(u, v) \geq f(v), \\ & a(u, u) = f(u), \end{cases} \tag{1.1.5}$$

if U is a closed convex cone with vertex 0, *or*

$$u \in U \quad \textit{and} \quad \forall v \in U, \quad a(u, v) = f(v), \tag{1.1.6}$$

if U is a closed subspace.

Proof. The projection u is completely characterized by the relations

$$u \in U \quad \text{and} \quad \forall v \in U, \quad a(\sigma f - u, v - u) \leq 0, \tag{1.1.7}$$

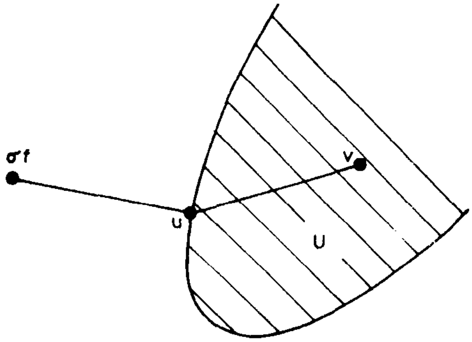

Fig. 1.1.1

the geometrical interpretation of the last inequalities being that the angle between the vectors $(\sigma f - u)$ and $(v - u)$ is obtuse (Fig. 1.1.1) for all $v \in U$. These inequalities may be written as

$$\forall v \in U, \quad a(u, v-u) \geqslant a(\sigma f, v-u) = f(v-u),$$

which proves relations (1.1.4).

Assume next U is a closed convex cone with vertex 0. Then the point $(u + v)$ belongs to the set U whenever the point v belongs to the set U (Fig. 1.1.2).

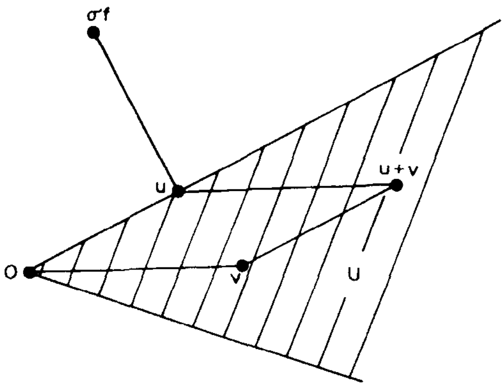

Fig. 1.1.2