METHODS AND APPLICATIONS OF NONLINEAR DYNAMICS

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Editor
A. W. SÁENZ
Naval Research Laboratory, Washington, D.C. 20375, U.S.A.
and
Catholic University, Washington, D.C. 20064, U.S.A.

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EDITOR'S FOREWORD

The present volume on *Methods and Applications of Nonlinear Dynamics* arose mainly from lectures given at the First International Course on Nonlinear Dynamics, which took place in Medellín, Colombia, on 1-5 September 1986. The aims of the Course were to discuss some of the fundamental theoretical ideas of modern nonlinear dynamics and their application to selected areas of physics, and also to help the participants to bridge the gap between textbook presentations and the contemporary research literature. The lectures were intended for and delivered to a Ph.D.-level audience composed of physicists and mathematicians. They were not primarily intended for experts, but rather for scientists interested in performing experimental or theoretical research on nonlinear dynamical phenomena occurring in real physical systems.

The volume opens with the survey lectures of Prof. A. F. Rañada (Universidad Complutense, Madrid) on the "Phenomenology of Chaotic Motion". They constitute a very readable introduction to chaotic dynamics and emphasize its physical aspects. The next contribution to the volume are the lectures of Prof. G. Turchetti (University of Bologna) on "Perturbative Methods for Hamiltonian Maps", which survey classical and recent contributions to this topic. In view of the importance of Hamiltonian dynamical systems in celestial mechanics, plasma physics, and many other areas, as well as the central importance of the contributions of contemporary Italian dynamicists to this wide subject, the inclusion of these lectures is more than justified. Prof. L. Vázquez (Universidad Complutense, Madrid) emphasizes the physical, intuitive aspects of nonlinear stochastic systems in his lectures on "An Elementary Introduction to Stochastic Processes", the next major topic discussed in the volume. The latter concludes with two survey articles by Prof. N. B. Abraham (Bryn Mawr College) and Prof. A. Zettl (University of California at Berkely) on "Chaos in Optical Systems" and "Chaos in Solid State Systems", respectively. These are two extremely active and fruitful areas of experimental and theoretical research in nonlinear dynamics in which there is active interest in Colombia and other Latin Americal countries.

Space limitations prevented the inclusion of survey articles on such important topics as chaos in fluid systems, although some salient phenomena in the latter domain are considered in Prof. Rañada's contribution to this volume.

I wish to express my deep appreciation to Profs. Rañada, Turchetti, Vázquez, and Zettl for generously contributing their time and expertise to the Medellín Course and to this book, as well as to Prof. Abraham (who for reasons beyond his control was unable to be in Medellín) for his contribution to the volume. The idea of the Course originated in conversations with Dr. S. M. Moore, whom I thank for many helpful suggestions. It was sponsored by the Asociación Pro-Centro Internacional de Física (ACIF) of Bogotá, Colombia, and Dr. E. Posada F. and Prof. G. Violini, the President and Executive Secretary, respectively, of ACIF, deserve thanks for their strong support of the meeting. My lifelong friend, Dr. Alberto Vásquez R., who was then governor of the Department of Antioquia, provided generous financial support. The Course was also sponsored by the Organization of American States. The National Organizing Committee was composed of Profs. E. Alvarez, J. Mahecha, and F. Medina (University of Antioquia), and Prof. R. Castañeda (National University of Colombia), to whom sincere thanks are due for their many efforts. Last but not least, I am grateful to Miss S. E. Dixon and Mrs. S. M. Montgomery of the Naval

Research Laboratory for the care with which they typed the relevant editorial changes to create a camera-ready manuscript, and to Mrs. H. S. Oxley of NRL for her valuable help with bibliographical matters.

It goes without saying that it is hoped that this volume will be useful to Latin American scientists and students, as well as those in the United States, Europe, and elsewhere interested in nonlinear physics.

A. W. Sáenz
Naval Research Laboratory
Washington, DC 20375
and
Physics Department
Catholic University
Washington, DC 20064

CONTENTS

PHENOMENOLOGY OF CHAOTIC MOTION

ANTONIO F. RAÑADA
Departamento de Física Teórica
Universidad Complutense
28040 Madrid, Spain

CONTENTS

CHAPTER 1 CHARACTERIZATION OF CHAOTIC MOTION

1.1 INTRODUCTION.

The realization that nature is much more complex and its behavior far richer than what was thought is one of the most appealing lessons of recent physics, since because of our apparent ability to predict with great precision the evolution of physical systems, our image of the world had become too simple, poor, and even cold. With the growing awareness that traditional physics gives only a first approximation to an essentially nonlinear world, a new perspective is emerging which approaches such seemingly unrelated problems as hydrodynamical turbulence, chemical kinetics, or celestial mechanics from the same point of view. In fact, it is found every day that systems which are described by very different models have many common traits and mathematical similarities which allow a unified treatment. To such an extent that it is possible to speak of universal patterns in the behaviour of nonlinear systems.

The most noteworthy rupture with traditional thought is the understanding that it is impossible, as a matter of principle, to predict precisely the long time behaviour of many systems because of the extreme instability of the solutions of their equations of motion. This was already known to Poincaré [1], who had studied this problem in his "Méthodes nouvelles de la mécanique céleste". But physics took the promising road of quantum theory and this question was forgotten, in spite of some warnings by Einstein.[2] Many years later, in 1954, it came again to light when Kolmogorov [3] stated what is now known as the KAM theorem (after his initial and those of Arnold [4] and Moser [5] who completed the proof), which characterizes the way in which instability arises when an integrable, and therefore regular, Hamiltonian is perturbed. In spite of its great importance, it applies only to Hamiltonian systems and not to the more abundant dissipative ones. The next development happened in 1963, when the meteorologist E.N. Lorenz [6)] found completely chaotic trajectories in a very simple dissipative system of three coupled ordinary differential equations. The great relevance of his result was not recognized until a decade later, but it is widely appreciated today and is often used as a paradigmatic example.

Such complex behavior is neither due to external noise, nor to a great number of degrees of freedom, nor to quantum effects. It just happens that trajectories, which in a certain moment are very close, separate exponentially in time and intermingle with others in an erratic and violent way. To designate this phenomenon, Lorenz coined the expresion *"butterfly effect"*, thus alluding to the sudden change in the initial conditions of the atmosphere because of the unexpected beating of the wings of a butterfly.

For classical physics all systems were, in principle, predictable,this idea being at the conceptual basis of mechanicism, which had such great importance in modern thought. But in the 19th century the study of systems with many degrees of freedom forced the beginning of a new probabilistic tradition, as their detailed study was thought to be impossible because of practical reasons. These two traditions were thought to be complementary, rather than opposite, the general attitude being the following: if a system has few degrees of freedom, use the techniques of Newtonian dynamics; if many, those of the new statistical mechanics. For this point of view, complex behaviour appears only when there are simultaneously many simple elements, so that systems with few coordinates must behave simply.

However, and to the surprise of most, even systems with only two degrees of freedom exhibit the butterfly effect or, in other words, their time evolution is turbulent and chaotic. This is called deterministic chaos, since the governing equations are deterministic but have solutions exhibiting stochastic properties. This discovery is causing a very deep change in our ideas on motion and, according to many, is bringing about a conceptual revolution which can be compared to those of relativity or quantum theory. Maxwell, one of the founding fathers of statistical mechanics, glimpsed it, when he wrote (quoted by {Berry, 1978})(curly brackets denote general references):

"If, therefore, those cultivators of the physical sciences. . . . . are led in pursuit of the arcana of science to the study of the singularities and instabilities, rather than the continuities and stabilities of things, the promotion of natural knowledge may tend to remove that prejudice in favor of determinism which seems to arise from assuming that the physical science of the future is a mere magnified image of that of the past".

In the case of Hamiltonian systems, it is possible to characterize in a simple way those which are regular, that is, nonchaotic. These are the integrable systems, so called because their equations of motion can be reduced to quadratures. Systems in this class have as many independent constants of the motion with vanishing Poisson brackets as they have degrees of freedom. On the other hand, those for which this is not the case, as happens in the famous three body problem, are called non-integrable and they exhibit chaotic behavior.

In these lectures we will not deal with such systems, but with the more abundant nonconservative ones, which have a richer and more diverse behavior. There is, in particular, a very important difference concerning stability. Conservative systems, contrary to dissipative ones, cannot have asymptotically stable solutions; however,this kind of stability is essential to many of the natural world structures, as this occurs in the important case of living beings.

1.2 CHAOS IS UBIQUITOUS

Since Lorenz's paper the list of systems in which chaos has been shown to exist, either experimentally or by the numerical solution of the equations of motion, grows at an accelerated pace. We can mention the following:

- Hamiltonian systems
- Celestial mechanics (e.g., the three-body problem)
- Fluids
- Lasers

- Nonlinear optical systems
- Solid state
- Particle accelerators
- Plasmas
- Chemical reactions
- Population dynamics
- Biological systems (e.g. the heart cell, the brain cell)

Let us consider some examples.

(a) The Forced Pendulum

The pendulum is frequently presented in elementary textbooks as one of the simplest physical systems. However, it shows chaotic behaviou if it is damped and is sumbmitted to a periodic force. [7,8] This example has great interest, because the equation of motion is the same as that of a Josephson junction submitted to microwave radiation:

$$\ddot{\varphi} + \frac{1}{\tau}\dot{\phi} + \omega_o^2 \sin\phi = \Gamma \cos \omega_d t, \qquad (1.1)$$

where ϕ is proportional to the potential difference, τ is the damping time, ω_d is the natural frequency and ω_d that of the microwave. For some values of Γ and ω_d there are chaotic solutions (see Fig. 1.1).

(b) The Sitnikov Case of the Three-Body Problem

That the three-body system is chaotic is shown very clearly in the particular case first studied by Sitnikov[9,10] (see also (Berry,1978)). As is seen in figure 1.2, two equal primaries of mass M follow elliptic orbits of excentricity ϵ, around their center of mass G in the plane π. A planetoid of negligible mass moves in their gravitatory field, along the straight line orthogonal to π at G.The equation of motion is

$$\ddot{z} = - \frac{z}{\{z^2 + \frac{1}{4}(1 - \cos 2t)^2\}^{3/2}} . \qquad (1.2)$$

If the orbits are circular, ϵ=0 and the system is integrable, the motion being periodic and consisting in a nonlinear oscillation around G.

If not, the motion is aperiodic and chaotic. If t_k are the values of time when the planetoid passes through the plane π, we can define the "periods" as $\tau_k = t_{k+1} - t_k$. It can be shown then that there exists a function $S(\epsilon)$, such that $S(\epsilon)\to\infty$ if $\epsilon\to 0$, with the property that for any random sequence $a_k > S(\epsilon)$,there is an initial condition $(z(0), \dot{z}(0))$ such that $\tau_k = a_k$; in other words there are infinite sets of solutions which can be characterized by random sequences of numbers. And this is so, in spite of the apparent simplicity of the system, which has only one degree of freedom.

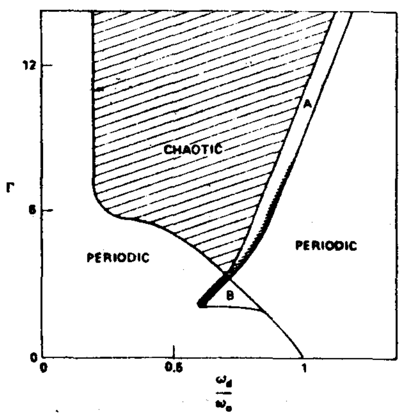

Figure 1.1. The forced pendulum (after Huberman et al., 1980 [7]).

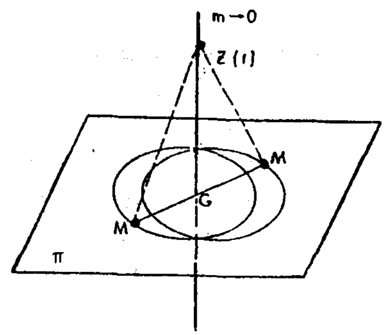

Figure 1.2. The Sitnikov case.

(c) Storage Rings

One of the systems in which chaos is posing difficult design problems is that of storage rings, in which charged particles move in electromagnetic fields((Helleman, 1980)) . If only one of the beams is travelling along a circular path, the transverse oscillations of the protons are described by the equation

$$\frac{d^2y}{d\phi^2} = - Q^2 y,$$

where y is the normal deviation, ϕ is the angular coordinate and Q is a characteristic frequency. The collisions with a second beam are practically instantaneous and can be modelled by

$$\frac{d^2y}{d\phi^2} = - Q^2 y + B\, F(y) \sum_n \delta(\phi - 2\pi n). \tag{1.3}$$

Although very few of the protons collide, all suffer a strong nonlinear force BF(y). The system (1.3) cannot be solved analitycally and its numerical treatment is very difficult, since in a typical experiment, the protons undergo more revolutions than those of by the Earth around the Sun in all its history. However, we can obtain a simpler discrete equation in the following way. Let us define

$$y_t = y(2\pi t^+) \ , \quad p_t = \frac{dy}{d\phi}(2\pi t^+),$$

$$C = \cos 2\pi Q, \qquad C = \sin 2\pi Q .$$

Equation (1.3) can then be written as

$$y_{t+1} = C\, y_t + (S/Q)\, p_t \ ,$$

$$p_{t+1} = -SQy_t + C\, p_t + BF(y_{t+1}) \ ,$$

and after eliminating p one has

$$y_{t+1} + y_{t-1} = 2C\, y_t + (BS/Q)\, F(y_t) \ , \tag{1.4}$$

an algebraic equation easy to solve. In Figure 1.3 we can see a typical diagram (y_{t+1}, y_t) for $F(y) = 2(1-\exp(-y^2/2))/y$, although many functions give rise to similar properties. The lower part is an enlargement around one of the hyperbolic points. All the chaotic points which "fill" a bidimensional region belong to the same orbit.

(d) The Rayleigh-Bénard Convection

This interesting phenomenon ((Bergé et al., 1984),(Schuster, 1984)) was discovered by Bénard in 1900 and explained by Lord Rayleigh in 1916. If a fluid layer is placed between two horizontal plates such that the lower one is hotter, the difference of the temperature being, there is a competition between

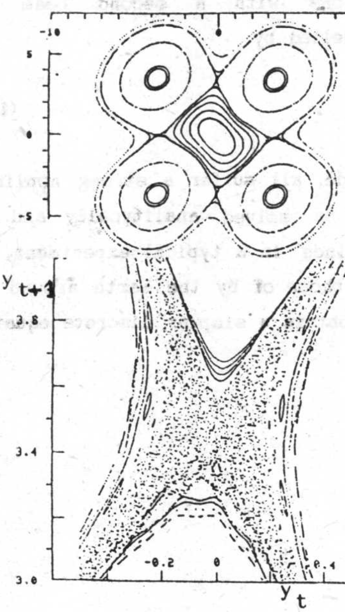

Figure 1.3. Phase space of Eq.1.4 (after Eminhizer, 1980 [11]).

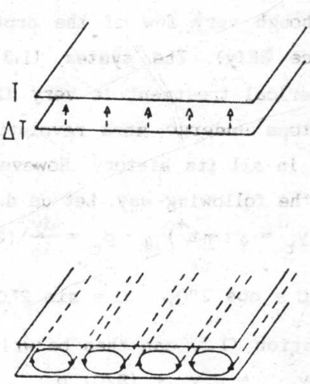

Figure 1.4. The Rayleigh-Bénard convection.

the tendency of the hotter fluid to rise and that of the cold one to fall (Figure 1.4). The viscosity is opposed to these tendencies and is able to impede any motion up to a certain threshold of ΔT. But, above this critical value, the equilibrium becomes unstable and the fluid begins to move with the classic convection rolls. The characteristic time for the equalization of the temperature by thermal diffusion is $\tau_T = d^2/D_T$, where d is the distance between the plates and D_T the thermal diffusivity. On the other hand, the characteristic time of the equalization through motion is $\tau_m = \eta/(\rho_0 g\alpha d\Delta T)$, where η is the viscosity, ρ_o the density, g the acceleration of the gravity and α the dilatation coefficient. In order that a permanent effect be produced, the ratio of these quantities, called the Rayleigh number, must be greater than a certain critical value R_o, that is,

$$Ra = \tau_T/\tau_m = \frac{\rho_o g\alpha d^3}{\eta D_T}\,\Delta T > Ro. \tag{1.5}$$

The Rayleigh number is the control parameter. In order to modify it, the experimenter changes the difference ΔT. Above R_o the convection rolls appear and above another critical value the regularity disappears and the motion does not follow any recognizable pattern.

(e) The Belousov-Zhabotinsky Reaction

In 1958, the Russian chemist Belousov observed an oscillating chemical reaction, made apparent by an alternation of the color of the solution, which changed between yellow and colorless (In 1921 Bray, in California, had made similar observations, but nobody paid attention to his work). Some years later, Zhabotinsky studied the phenomenon in detail in his doctoral thesis. In the so called Belousov-Zhabotinsky reaction, an organic molecule (malonic acid) is oxidized by bromate ions, the process being catalyzed by a redox system (Ce^{4+}/Ce^{3+})[12,13] (see also {Schuster, 1984}). The basic reactants are

$$Ce_2(SO_4)_3,\ NaBrO_3,\ CH_2(COOH)_2,\ H_2SO_4,$$

to which a color indicator is added. The reaction is very complex, with as many as 18 elementary steps, and depends on several control parameters such as the temperature and the mean residence time in the open reactor. The variables of the associated dynamical system are the concentrations of the reactants. It is usual to observe that of the ion Ce_4^+, this being specially easy because of its strong absorption of light of 340nm. For some values of the control parameter, the concentration of Ce_4^+ (and that of the other chemicals) oscillates periodically. But above a certain critical value the behaviour is chaotic.

(f) The Bernoulli Shift

The Bernoulli shift[14] (see also {Arnold and Avez}, 1978) is a discrete dynamical system in [0,1), defined as

$$x_{n+1} = \sigma(x_n) = 2\,x_n \pmod 1. \qquad (1.6)$$

It is convenient to use the binary representation of x:

$$x_o = 0.a_1a_2a_3 \ldots a_n \ldots = \sum_1^\infty a_k 2^{-k},$$

since it is then clear that the action of σ consists in erasing the first digit

$$\sigma(0.a_1a_2\dots) = 0.a_2a_3a_4\dots .$$

That of σ^k is therefore the elimination of the first k digits. This strongly chaotic system has the following properties:
(i) It has sensitive dependence on the initial conditions, because the errors double at each step. If the distance between two "seeds" x_o and x_o' is smaller than $1/2^n$, they have the first n digits in common, but they differ in the (n+1). Consequently the sequences generated by them will be completely different after the iteration number (n+1). If we only know the first n digits of x_o we know nothing about x_{n+1}.
(ii) Let us consider the coarse-grained evolution based in the partition of the phase space into the two intervals $L\equiv[0,½)$, $R\equiv[½,1)$. To every trajectory x_n we associate a sequence of L's and R's, depending on where x_n is. It is certainly isomorphic to the binary representation of x_o. For instance to 0.1001011... there corresponds RLLRLRR... If we identify R with heads and L with tails, x_o is equivalent to tossing a coin an infinite number of times. In fact, as none of the two sequences, digits of x_o or heads or tails, follow in general a regular pattern, if we are given a sequence of R's and L's, we cannot tell how it has been obtained.
(iii) The system is ergodic. This means that the trajectory generated by any seed, with the exception of a set of measure zero (the rationals), passes arbitrarily close to any number x in [0,1). To be specific, if we require that it passes closer than $\epsilon=2^{-n}$, it is necessary that there is an integer m, such that the first n digits of x_m coincides with those of x. But this is precisely the case, since the decimal representation of any irrational contains any finite sequence of digits an infinite number of times.

(iv) In order to predict the sequence x_k for all k it is necessary to know the initial condition x_o with an infinite number of digits, that is with zero error.This is precisely the reason for the unpredictability of a chaotic system: the determinism of the equation of motion does not allow an effective prediction of the trajectory if the initial conditions are not known with infinite precision; but this is impossible in data obtained by measurement or calculation, as this would imply an infinite amount of information.

It is easy to understand the mechanism which is responsible for the chaos. It consists in the two effects which appear schematically represented in figure 1.6:

(a) Because of the product by 2, the intervals are "**stretched**", the errors are amplified and the neighbouring points are separated.

(b) Because only the decimal part is taken, the stretched total interval is "**folded**" and the previously separated points are intermingled.

This combination of stretching and folding is similar to the shuffling of a deck of cards. In fact, in this case, the deck is first expanded, so that the cards are slightly separated, and is later folded on itself. For this reason this mechanism, which is present in general chaotic systems, could be called shuffling effect. Curiously enough, many natural deterministic systems behave as cards which are shuffled.

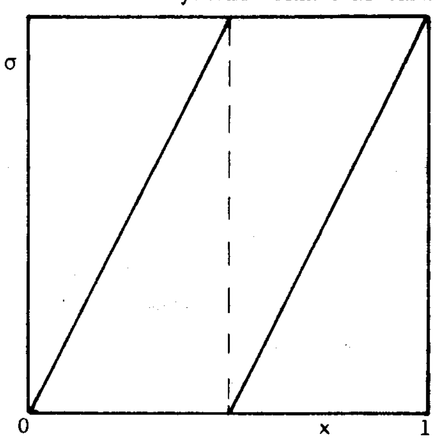

Figure 1.5. The Bernouilli shift $\sigma(x)$.

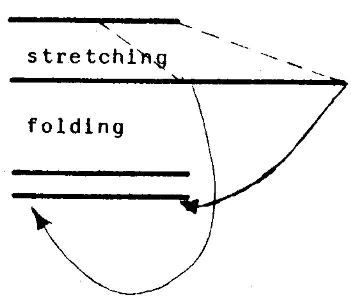

Figure 1.6. Stretching and folding in the Bernouilli shift.