Lagrangian Analysis and Quantum Mechanics

Mathematical Structure Related to
symptotic Expansions and the Maslov Index

Jean Leray

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A Mathematical Structure Related to Asymptotic Expansions and the Maslov Index

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Preface

Only in the simplest cases do physicists use exact solutions, $u(x)$, of problems involving temporally evolving systems. Usually they use asymptotic solutions of the type

$$u(v,x) = \alpha(v,x)e^{v\varphi(x)}, \tag{1}$$

where

- the *phase* φ is a real-valued function of $x \in X = \mathbf{R}^l$;
- the *amplitude* α is a formal series in $1/v$,

$$\alpha(v,x) = \sum_{r=0}^{\infty} \frac{1}{v^r}\alpha_r(x),$$

whose coefficients α_r are complex-valued functions of x;
- the *frequency* v is purely imaginary.

The differential equation governing the evolution,

$$a\left(v,x,\frac{1}{v}\frac{\partial}{\partial x}\right)u(v,x) = 0, \tag{2}$$

is satisfied in the sense that the left-hand side reduces to the product of $e^{v\varphi}$ and a formal series in $1/v$ whose first terms or all of whose terms vanish. The construction of these asymptotic solutions is well known and called the WKB method:

- The phase φ has to satisfy a first-order differential equation that is nonlinear if the operator a is not of first order.
- The amplitude α is computed by integrations along the characteristics of the first-order equation that defines φ.

In quantum mechanics, for example, computations are first made as if

$$v = \frac{i}{\hbar} = \frac{2\pi i}{h}, \qquad \text{where } h \text{ is Planck's constant,}$$

were a parameter tending to $i\infty$; afterwards v receives its numerical value v_0.

Physicists use asymptotic solutions to deal with problems involving equilibrium and periodicity conditions, for example, to replace problems of wave optics with problems of geometrical optics. But φ has a jump and α has singularities on the envelope of characteristics that define φ: for example, in geometrical optics, α has singularities on the caustics, which are the images of the sources of light; nevertheless geometrical optics holds beyond the caustics.

V. P. Maslov introduced an index (whose definition was clarified by *I. V. Arnold*) that described these phase jumps, and he showed by a convenient use of the Fourier transform that these amplitude singularities are only apparent singularities. But he had to impose some "quantum conditions." These assume that ν has some purely imaginary numerical value ν_0, in contradiction with the previous assumption about ν, namely, that ν is a parameter tending to $i\infty$. The assumption that ν tends to $i\infty$ is necessary for the Fourier transform to be pointwise, which is essential for Maslov's treatment. *A procedure, avoiding that contradiction and guided by purely mathematical motivations, that makes use of the Fourier transform, expressions of the type* (1), *Maslov's quantum conditions, and the datum of a number ν_0 does exist*, but no longer tends to define a function or a class of functions by its asymptotic expansion. It leads to a *new mathematical structure*, ***lagrangian analysis***, which requires the datum of a constant ν_0 and is based on symplectic geometry. Its interest can appear only a posteriori *and could be quantum mechanics*. Indeed this structure allows a new interpretation of the *Schrödinger*, *Klein-Gordon*, and *Dirac* equations provided

$$\nu_0 = \frac{i}{\hbar} = \frac{2\pi i}{h}, \qquad \text{where } h \text{ is Planck's constant.}$$

Therefore the real number $2\pi i/\nu_0$ whose choice defines this new mathematical structure can be called *Planck's constant*.

The introductions, summaries, and conclusions of the chapters and parts constitute an abstract of the exposition.

Historical note. In Moscow in 1967 *I. V. Arnold* asked me my thoughts on Maslov's work [10, 11]. The present book is an answer to that question.

It has benefited greatly from the invaluable knowledge of *J. Lascoux*.

It introduces ν_0 for defining lagrangian functions on V (chapter II, §2, section 3) in the same manner as Planck introduced $\hbar$ for describing the spectrum of the blackbody. Thus the book could be entitled

The Introduction of Planck's Constant into Mathematics.

January 1978
Collège de France
Paris 05

Index of Symbols

A	I,§1,definition 1.2*
C	field of the complex numbers; $\dot{\mathbf{C}} = \mathbf{C}\setminus\{0\}$
$\mathbf{E}^3$	3-dimensional euclidean space
$\mathbf{I} = i[1, \infty[$	II,§1,1
N	set of the natural numbers (i.e., integers $\geqslant 0$)
R	field of the real numbers
$\mathbf{R}_+$	set of the real positive numbers
$\mathbf{S}^n$	n-dimensional sphere
Z	ring of the integers
A	element of **A**: I; II
B	bounded set: I; II
A, B, C	functions of M, coefficients of the Schrödinger–Klein-Gordon operator: III,§1,example 4
E	neutral element of a group: I; II
	atomic energy level: III; IV
F	any function: II
	the function in III,§1,(4.23)
G	group: I; II
	function: III; IV
H	hamiltonian: II,§3,1; II,§3,definition 6.1
Hess	hessian: I,§1,definition 2.3
I_k, J_k	elements of $\mathbf{E}^3$: III,§1,1
Inert	index of inertia: I,§1,2; I,§2,4; I,§2,definition 7.2

*Each chapter (I, II, III, IV) is divided into parts (§1, §2, §3, ...), which in turn are divided into sections (0, 1, 2, ...). References to elements of sections (for example, theorems, equations, definitions) in the same chapter, part, and section are by one or two numbers: in the latter case the first number refers to the section and is followed by a period. References to elements of sections in another chapter, part, and/or section are by a string of numbers separated by commas. For example, a reference *in chapter I, §2, section 3* to the one theorem in this chapter, part, and section is simply theorem 3; to the one theorem in this chapter and part but section 4 is theorem 4; to the one theorem in this chapter but §3 of section 4 is §3,theorem 4; and to the one theorem in chapter II, §3, section 4 is II,§3,theorem 4. Similarly, a reference *in chapter I, §2, section 3* to the first definition (of more than one) in chapter II, §3, section 4 is II,§3,definition 4.1.

$J^{(k)}$	matrices: II,§4,3; IV,§1,1
K	characteristic curve: II,§3,definition 3.1; III,§1,(2.14)
	function: III,§1,4
L, M, P, Q, R	functions: III,§1,1
N	function of (L, M): III,§1,(2.9); N_L and N_M are its derivatives
R (I, II), R_0 (III, IV)	frame in symplectic geometry: I,§3,2; I,§3,3
$\mathrm{Sp}(l)$	symplectic group: I,§1,definition 1.1
$\mathrm{Sp}_2(l)$	the covering group, of order 2, of the symplectic group: I,§1,definition 2.1
S	element of $\mathrm{Sp}_2(l)$
T	torus: III,§1,3
U	lagrangian function on V: II,§2,3
$\check{U}_R$	formal functions on $\check{V}$: II,§2,2
$U(l)$	unitary group: I,§2,2
V	lagrangian manifold: I,§2,9; I,§3,1
W	hypersurface of Z
$W(l)$	subset of $U(l)$: I,§2,lemma 2.1
X, Z	spaces: I,§1,1; I,§3,1
a	differential, formal or lagrangian operator: I,§1,definition 3.1; II,§1,definition 6.2; II,§2, definition 1.1
arg	argument
d	differentiation
$d^l x$	Lebesgue measure
det	determinant
e	2.71828...
	neutral element of a group: I
i	$\sqrt{-1}$
i_ξ	interior product: II,§3,2
j	quantum number: IV,§1,example 2
l	dimension of X: I; II (dimension of $X = 3$ in III, IV)

l, m, n	quantum numbers: III; IV
m, m_R	Maslov index: I,§1,definition 1.2; I,§1,(2.15); I,§2,definition 5.3; I,§2,theorem 5; I,§3,theorem 1; I,§3,3
s	element of Sp(l): I; II
	function: III; IV
t	function: II,§3,(3.10); II,§3,(3.13); III,§1,(2.6)
u	element of $U(l)$: I
${}^tu$	transpose of u: I,§2,2
u	formal number or function: II,§1,2
w	element of $W(l)$
x, y	elements of X
z	elements of Z
$\mathscr{A}$	I,§1,1
$\mathscr{B}, \mathscr{C}$,	II,§1,1
$\mathscr{F}, \mathscr{F}'$	space of formal or lagrangian functions or distribution: II,§1,2; II,§1,7; II,§2,2; II,§2,3; II,§2,5
$\mathscr{H}$	Hilbert space: I,§1,1
$\mathscr{L}$	Lie derivative: II,§3,definition 3.2
$\mathscr{N}$	neighborhood
$\mathscr{S}, \mathscr{S}'$	Schwartz spaces: I,§1,1
Γ	arc: I,§2
	curve: III,§1,(2.5)
Δ	laplacian (Δ_0 is the spherical laplacian): III,§3,(2.4)
Λ	lagrangian grassmannian: I,§2,2; I,§3,1
$\wedge$	exterior product
Π	projection: II,§1,theorem 2.1
Σ	apparent contour, Σ_{Sp}: I,§1,definition 1.3
	Σ_V: I,§2,9
	Σ_R: I,§3,2
Φ, Ψ, Θ	Euler angles: III,§1,(1.11); III,§1,(1.12)
Ω	open set in Z: II,§1,6; II,§2,1

	function: III,§1,(2.8)
Ω^6	open set in $\mathbf{E}^3 \oplus \mathbf{E}^3$: III,§1,1
α	amplitude: II,§1,2
β_0	lagrangian amplitude: II,§2,theorem 2.2
γ	arc or homotopy class
η, η_V	invariant measure of V: II,§3,definition 3.2
κ	characteristic vector: II,§3,definition 3.1
λ	element of Λ: I; II
λ, μ	functions: III,§1,(2.10)
ν	element of $\mathbf{I}$: II,§1,1
ν_0	$i/\hbar = 2\pi i/h$ ($h \in \mathbf{R}_+$): II,§2,3; II,§3,6
π	3.14159...
π_j	jth homotopy group: I,§2,3
$\sigma[\cdot], \sigma_j$	Pauli matrices: IV,§1,(1.6); IV,§1,(1.7)
χ	$\eta/d^l x$
φ	phase: I,§2,9; I,§3,2; II,§1,2
ψ	lagrangian phase: I,§3,1
ω, ϖ	pfaffian forms
ω_j	III,§1,(1.7)

Atomic Symbols: III; IV; passim (see III,§1,4, Notations)

E	energy
c	speed of light
h	Planck's constant
$\mathscr{A}_j$	potential vector
$\mathscr{H}$	magnetic field
α	1/137
β	Bohr magneton
ε	charge
μ	mass

Index of Concepts

amplitude	α
asymptotic class	$\mathscr{C}$
characteristic curve	K
characteristic vector	κ
energy	E
Euler angles	Φ, Ψ, Θ
formal number, functions	$u, \check{U}_R$
frames	$R; (I_1, I_2, I_3); (J_1; J_2; J_3)$
groups	$\mathrm{Sp}(l); U(l)$
hamiltonian	H
hessian	Hess
homotopy	π_j
index of inertia	Inert
interior product	i_ξ
lagrangian amplitude	β_0
lagrangian function	U
lagrangian manifold	V
lagrangian operator	a
lagrangian phase	ψ
Lie derivative	$\mathscr{L}$
Maslov index	m
matrix	$J^{(k)}, \sigma$
operator	a
Planck's constant	$\nu_0 = i/\hbar$
quantum numbers	l, m, n, j
spaces	$X, Z, \mathscr{F}. \mathscr{F}', \mathscr{H}, \mathscr{S}, \mathscr{S}'$
symplectic space	Z

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I The Fourier Transform and Symplectic Group

Introduction

Chapter I explains the connection between two very classical notions: the Fourier transform and the symplectic group.

It will make possible the study of asymptotic solutions of partial differential equations in chapter II.

§1. Differential Operators, the Metaplectic and Symplectic Groups

0. Introduction

Historical account. The metaplectic group was defined by *I. Segal* [14]; his study was taken up by *D. Shale* [15]. *V. C. Buslaev* [3, 11] showed that it made Maslov's theory independent of the choice of coordinates. *A. Weil* [18] studied it on an arbitrary field in order to extend *C. Siegel's* work in number theory.

Summary. We take up the study of the metaplectic group in order to specify its action on $\mathscr{S}(\mathbf{R}^l)$, $\mathscr{H}(\mathbf{R}^l)$, and $\mathscr{S}'(\mathbf{R}^l)$ (see theorem 2) and its action on differential operators (see theorem 3.1).

1. The Metaplectic Group $\mathrm{Mp}(l)$

Let X be the vector space $\mathbf{R}^l$ ($l > 1$) provided with *Lebesgue measure* d^lx. Let X^* be its dual, and let $\langle p, x\rangle$ be the value obtained by acting $p \in X^*$ on $x \in X$.

Spaces of functions and distributions on X. The *Hilbert space* $\mathscr{H}(X)$ consists of functions $f: X \to \mathbf{C}$ satisfying

$$|f| = \left(\int_X |f(x)|^2 d^lx\right)^{1/2} < \infty.$$

The *Schwartz* space $\mathscr{S}(X)$ [13] consists of infinitely differentiable, rapidly decreasing functions $f: X \to \mathbf{C}$. That is, for all pairs of l-indices (q, r)

$$|f|_{q,r} = \operatorname*{Sup}_X \left|x^q\left(\frac{\partial}{\partial x}\right)^r f(x)\right| < \infty.$$

The topology of $\mathscr{S}(X)$ is defined by a *countable* fundamental system of neighborhoods of 0, each depending on a pair of l-indices (q, r) and a rational number $\varepsilon > 0$ as follows:

$$\mathcal{N}(q, r, \varepsilon) = \{f \mid |f|_{q,r} \leqslant \varepsilon\}.$$

The bounded sets B of $\mathscr{S}(X)$ are thus all subsets of bounded sets of $\mathscr{S}(X)$ of the following form:

$$B(\{b_{q,r}\}) = \{f \mid |f|_{q,r} \leqslant b_{q,r}\ \forall q, r\}, \qquad q, r \in \mathbf{N}^l, \quad b_{q,r} \in \dot{\mathbf{R}}_+.$$

The *Schwartz space* $\mathscr{S}'(X)$ is the dual of $\mathscr{S}(X)$ [13]; its elements are the *tempered distributions*: such an element f' is a continuous linear functional

$$\mathscr{S}(X) \to \mathbf{C}.$$

The value of f' on f will be denoted by $\int_X f'(x)\, f(x)\, d^l x$, although the value of f' at x is not in general defined. The bound of f' on a bounded set B in $\mathscr{S}(X)$ is denoted by

$$|f'|_B = \operatorname*{Sup}_{f \in B} \left| \int_X f'(x)\, f(x)\, d^l x \right|.$$

The continuity of f' is equivalent to the condition that f' is *bounded*: $|f'|_B < \infty\ \forall B$. The topology of $\mathscr{S}'(X)$ is defined by a fundamental system of neighborhoods of 0, each depending on a bounded set B of $\mathscr{S}(X)$ and a number $\varepsilon > 0$, as follows:

$$\mathcal{N}'(B, \varepsilon) = \{f' \mid |f'|_B \leqslant \varepsilon\}.$$

Unlike the above, this topology cannot be given by a countable fundamental system of neighborhoods of zero.

Let us recall the following theorems. $\mathscr{H}(X)$ can be identified with a subspace of $\mathscr{S}'(X)$:

$$\mathscr{S}(X) \subset \mathscr{H}(X) \subset \mathscr{S}'(X).$$

The *Fourier transform* is a continuous automorphism of $\mathscr{S}'(X)$ whose restrictions to $\mathscr{H}(X)$ and $\mathscr{S}(X)$ are, respectively, a unitary automorphism and a continuous automorphism.

$\mathscr{S}(X)$ is dense in $\mathscr{S}'(X)$.

For the proof of the last theorem, see L. Schwartz [13]: chapter VII, §4, the commentary on theorem IV, and chapter III, §3, theorem XV; alternatively, see chapter VI, §4, theorem IV, theorem XI and its commentary.

Differential operators associated with elements of $Z(l) = X \oplus X^*$. Let v be an imaginary number with argument $\pi/2$: $v/i > 0$.

Let a^0 be a linear function, $a^0: Z(l) \to \mathbf{R}$. Let $a^0(z) = a^0(x, p)$ be its value at $z = x + p\,[z \in Z(l), x \in X, p \in X^*]$. The operator

$$a = a^0\left(x, \frac{1}{v}\frac{\partial}{\partial x}\right)$$

is a *self-adjoint* endomorphism of $\mathscr{S}'(X)$: the adjoint of a, which is an endomorphism of $\mathscr{S}(X)$, is the restriction of a to $\mathscr{S}(X)$. The operators a and the functions a^0 are, respectively, elements of two vector spaces $\mathscr{A}$ and $\mathscr{A}^0$. These spaces are both of dimension $2l$ and are naturally isomorphic:

$$\mathscr{A}^0 \ni a^0 \mapsto a \in \mathscr{A}.$$

We say that a is the differential operator associated to $a^0 \in \mathscr{A}^0$. By (1.2), $\mathscr{A}^0$, which is the dual of $Z(l)$, will be identified with $Z(l)$.

The commutator of a and $b \in \mathscr{A}$ is

$$[a, b] = ab - ba \in \mathbf{C};$$

$c \in \mathbf{C}$ denotes the endomorphism of $\mathscr{S}'(X)$:

$$c: f \mapsto cf \qquad \forall f \in \mathscr{S}'(X).$$

In order to study this commutator, we give $Z(l)$ the *symplectic structure* $[\cdot,\cdot]$ defined by

$$[z, z'] = \langle p, x'\rangle - \langle p', x\rangle,$$

where $z = x + p$, $z' = x' + p'$, x and $x' \in X$, and p and $p' \in X^*$.

Each function $a^0 \in \mathscr{A}^0$ is defined by a unique element a^1 in $Z(l)$ such that

$$a^0(z) = [a^1, z]. \tag{1.1}$$

This gives a natural isomorphism

$$Z(l) \ni a^1 \mapsto a^0 \in \mathscr{A}^0. \tag{1.2}$$

The commutator of a and $b \in \mathscr{A}$ is clearly