ELEMENTS OF MATHEMATICAL LOGIC

The Elements of Mathematical Logic

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PREFACE

This book is intended for readers who, while mature mathematically, have no knowledge of mathematical logic. We attempt to introduce the reader to the most important approaches to the subject, and, wherever possible within the limitations of space which we have set for ourselves, to give at least a few nontrivial results illustrating each of the important methods for attacking logical problems. Since Lewis' SURVEY OF SYMBOLIC LOGIC and Jørgensen's TREATISE ON FORMAL LOGIC, both of which are now obsolete, the only work of this nature has been the excellent book of Church, which is not suitable for beginners and which is not easily accessible. Thus the present book differs from those which confine themselves to the detailed development of one particular system of formal logic. We have emphasized instead the modern tendency of analyzing the structure of a system as a whole. We feel that too many authors in this field have overlooked the necessity of exhibiting the power of logical methods in non-trivial problems. Otherwise mathematical logic is a mere shorthand for transcribing results obtained without its aid, not a tool for research and discovery.

Thus in the chapter on the logic of classes we have a section on the structure and representation of Boolean algebras, which is applied in the next chapter to the study of deductive systems. In the third chapter we sketch the methods of Russell, Quine, Zermelo, Curry, and Church for the construction of logics of propositional functions. Finally, we give a brief introduction to the general syntax of language, with applications to undecidability and incompleteness theorems.

We have attempted to make the exposition as elementary as possible throughout. Nevertheless, those who are unfamiliar with modern algebra may find it advisable to skip the proofs in Chapter I, Section 3, on the first reading.

In the last chapter we use the profound and beautiful ideas of Post. We hope that one by-product of this book will be a more widespread recognition and appreciation of his work, which amounts to the creation of a new branch of mathematics of the same fundamental importance as algebra and topology.

The connoisseur may find of some interest (1) the insistence on the demonstrable properties of a formal system as a criterion for its acceptability, (2) the simple proof of the completeness of the theory of combinators,* (3) the simple explicit example of a recursively unsolvable problem in elementary number theory, (4) the first connected exposition of all the essential steps in the proof of Church's theorem on the recursive unsolvability of the decision problem for the restricted function calculus.

Much of the material was presented in a course given by the author at Lund University, Sweden, in the spring of 1948.

It is impossible for me to express adequately my debt to the late Professor H. B. Smith for his constant kindness and generosity. I am grateful to Professors Churchman, Post, Curry, McKinsey, Huntington, and Stone for their friendly encouragement when I was beginning my mathematical career. I cannot refrain from also thanking Professors Cohen and Nagel, since it was a misinterpretation of a footnote in their book which led me to abandon chemistry for mathematics twelve years ago! I thank Dover Publications, Inc. for its unfailing courtesy and helpfulness during the preparation of this book. Finally, I should like to express my gratitude to my beloved wife, Elly, for providing the stimulus and the working conditions without which the book could not have been written.

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*Curry has arrived independently at essentially the same simplification of the theory of combinators. This appeared since the above was written in Synthèse, Vol. VII, 1948–49, No. 6-A, p. 391–398.

INTRODUCTION

In this book we shall study the laws of logic by mathematical methods. This may seem unfair, since logic is used in constructing mathematical proofs, and it might appear that the study of logic should come before the study of mathematics. Such a procedure is, however, typical of science. Our actual knowledge is a narrow band of light flanked on both sides by darkness. We may, on the one hand, go forward and develop further the consequences of known principles. Or else we may press backward the obscurity in which the foundations of science are enveloped. Just by using mathematical methods, i.e. by working with ideograms (symbols for ideas) instead of ordinary words (symbols for sounds), we can throw new and important light on the logical principles used in mathematics. This approach has led to more knowledge about logic in one century than had been obtained from the death of Aristotle up to 1847, when Boole's masterpiece was published.

We begin with the simplest branch of the subject, the logic of classes. After an informal introduction, in which we derive the properties of classes by a free use of naive intuition, we formulate that theory as a deductive science, that is, as a science in which the assumptions are explicitly stated, and in which everything else follows from the assumptions by means of explicitly stated rules. The assumptions are stated in terms of certain notions which are not analyzed further and are taken as undefined. All other concepts of the science are defined in terms of these.

We then proceed to a study of the system as a whole. That is, instead of developing more and more consequences of the assumptions, we try to find general characteristics of the science itself. This is typical of the modern tendency to emphasize the *structure* of a science, to derive theorems *about* the science, rather than to concentrate on the detailed derivation of results

within the science. This study of the structure of the logic of classes culminates in Chapter I, Section 3.

We then apply the same methods to the logic of propositions. In doing this, we uncover a striking similarity between this science and the logic of classes. It is precisely through formulating these logics as deductive sciences that we see that both are special examples of a general theory.

The logic of propositions has been the subject of much controversy among logicians and mathematicians. We discuss the various alternative approaches which have been proposed.

We then try to construct general logical theories which are adequate for at least a large part of mathematics. Here we run into difficulties since the unreined use of naive intuitive reasoning leads to devastating paradoxes. Thus we must seek a theory which admits as much as possible of the reasoning intuitively accepted as valid, but includes such restrictions as to evade the paradoxes. But a profound theorem of Gödel shows that no logical theory of a very general type can include methods of reasoning strong enough for the proof of its own consistency. Indeed, in any system of logic of this general type, there are propositions which can be proved by an argument outside the system but which cannot be proved within the system. Thus no formal logical system of this type, which includes all adequate logics so far proposed, can contain all valid modes of reasoning. All that we can hope for is stronger and stronger systems which are adequate for more and more powerful arguments, or else some system radically different from anything so far proposed.

In order to arrive at such results as Gödel's, it is necessary for us to scrutinize our tools more closely. In a deductive science the undefined terms are denoted by certain symbols, which may be blobs of printer's ink, speech sounds, printed marks representing the latter, etc. The propositions of this science are communicated by means of these signs. These signs, together with the rules governing their use and combination, constitute a language for stating relationships within the science. This is called the *object language*. In an exposition of the science the assumptions must be communicated in a language whose mean-

ing is already assumed to be known, say English. This is called the *syntax language*. We use the object language to talk *within* the science and the syntax language to talk *about* the science. In ordinary usage the confusion between the two leads to no difficulty, but when the science under consideration is logic itself we must lean over backwards to avoid unclarity.

The primitive signs of the object language are called its alphabet. Certain combinations of these signs may be assigned meanings. Such combinations are often called words or sentences. If a certain combination of signs denotes an object, then this combination will be a name for that object. In speaking about the object we use a name for it. Thus "Dewey smiled" is a sentence wherein we mention the man Dewey by using his name, the word "Dewey." When we are talking about a name or a symbol, it is convenient to use a specimen enclosed in quotation marks as a name of the name or symbol. Thus, " "Dewey" " is a name of "Dewey," which is, in turn, a name of Dewey, who is a man. Again, on p. 2, 25th line from the bottom, we are speaking about a name of the universal class, while on the next line it is the null class itself which is mentioned. To avoid the use of names of names of names and the like, we shall also use such phrases as "the letter —" or "the sign —" as names of the symbols of which specimens are exhibited. It is often overlooked that while we cannot put a man on the printed page and are thus *forced* to use a name when writing about him, we do have greater resources when we wish to write about symbols.

In particular, a sentence is a name of a proposition. We shall say that the sentence *expresses* the proposition, and we shall often use "statement" as a synonym for "sentence." We shall often use the phrase "the proposition that p" to indicate the proposition expressed by "p." Careful attention to these matters helps in discussing ticklish questions.

We are thus led, in chapter IV, to the mathematical analysis of language. Whereas in the previous chapters our attention is centered on the relationships expressed by the object language, in the last chapter we focus our attention on the structure of the language apart from its meaning. The former process is some-

times called the semantical study of language, i.e. the study of the meanings expressed by the language, while the latter is often called the syntactical study of the language. The methods we use were developed especially by Post. We find in this chapter that certain classes of languages, which include practically all languages which have been precisely formulated, can be singled out and possess important common properties. It is exactly the mathematical method of abstracting from the special features of particular languages which enables us to prove rigorously a number of profound general truths, where metaphysicians would argue back and forth for centuries without ever reaching a conclusion which could be tested.

Mathematical logic is, then, no mere shorthand for expressing in ideograms what has already been discovered by reasoning in ordinary language. It is, rather, a powerful and versatile tool for solving problems which are inaccessible to other methods.

In the following we shall make references thus:

III2 denotes section 2 of chapter III;
T2 denotes theorem 2 of the present section;
T5.2.3 denotes theorem 3 of section 2 of chapter V;
[27]4 denotes number 4 by author 27 in Church's Bibliography, J. Symbolic Logic, vol. 1, no. 4;
[II]35 denotes the article beginning or reviewed on page 35 of vol. II, J. Symbolic Logic.

CONTENTS

Chapter I

THE LOGIC OF CLASSES

SECTION 1 INFORMAL INTRODUCTION

FUNDAMENTAL THEOREMS

Logic is the science of the valid processes of reasoning. In mathematical logic we investigate these processes by mathematical methods. In this first chapter we shall study the simplest branch of this science, the logic of classes.

For the moment we shall not attempt to analyze the concept of "class." Rather we shall take it as undefined but shall assume that its intuitive meaning is known. By a class we shall mean any collection of things, for example, the class of all men or the class of red-headed baboons. The members of the class may be abstractions or may be in some other sense not tangible; thus the class of positive integers and the class of jabberwockies are perfectly good classes. We shall denote classes by small Greek letters.

We shall say that the class α is the same as the class β if and only if they have exactly the same members. Thus the class of even primes is the same as the class whose only member is the number 2. We shall denote the relationship "α is the same as β" by the symbols "$\alpha = \beta$." The following propositions are evident:

T1. $\alpha = \alpha$;
T2. *if* $\alpha = \beta$, *then* $\beta = \alpha$;
T3. *if* $\alpha = \beta$ *and* $\beta = \gamma$, *then* $\alpha = \gamma$.

In most statements, if $\alpha = \beta$, then "α" may be substituted for "β" at any point without changing the truth or falsity of the statement.

We shall symbolize the statement "x is a member of α" by "$x \in \alpha$."

If α and β are classes, we shall denote by "$\alpha \cap \beta$" the class of all objects which are members of both α and β. Similarly, we shall use "$\alpha \cup \beta$" for the class of all things which are either in α or in β or in both. Thus if α is the class of females and β is the class of engineers, then $\alpha \cap \beta$ is the class of female engineers, and $\alpha \cup \beta$ is the class of all objects which are either females or engineers or both. By α' we shall mean the class of all objects which are not in α. The class $\alpha - \beta = \alpha \cap \beta'$, by definition, so that $\alpha - \beta$ is the class of all objects which are in α but not in β. Two special classes are of importance, the universal class, denoted by "1", which is the class containing all things, and the null class, 0, which is the class which has no members.

These symbols have been introduced so that we may construct an algebra of classes. They enjoy the following properties:

T4. $\alpha \cap \beta = \beta \cap \alpha$;
T5. $\alpha \cap (\beta \cap \gamma) = (\alpha \cap \beta) \cap \gamma$;
T6. $\alpha \cup \beta = \beta \cup \alpha$;
T7. $\alpha \cup (\beta \cup \gamma) = (\alpha \cup \beta) \cup \gamma$;
T8. $\alpha \cap \alpha = \alpha \cup \alpha = \alpha$;
T9. $\alpha \cap (\beta \cup \gamma) = (\alpha \cap \beta) \cup (\alpha \cap \gamma)$;
T10. $\alpha \cup (\beta \cap \gamma) = (\alpha \cup \beta) \cap (\alpha \cup \gamma)$;
T11. $\alpha \cup \alpha' = 1$;
T12. $\alpha \cap \alpha' = 0$;
T13. $\alpha \cap 1 = \alpha \cup 0 = \alpha$;
T14. $\alpha \cup 1 = 1$;
T15. $\alpha \cap 0 = 0$;
T16. $(\alpha')' = \alpha$;
T17. $0' = 1; 1' = 0$;
T18. $(\alpha \cup \beta)' = \alpha' \cap \beta'$;
T19. $(\alpha \cap \beta)' = \alpha' \cup \beta'$;
T20. $\alpha \cup (\alpha \cap \beta) = \alpha \cap (\alpha \cup \beta) = \alpha$.

These propositions are for the most part obvious. Thus T4 says that if x is in $\alpha \cap \beta$, i.e. if x is in both α and β, then x is in $\beta \cap \alpha$, and conversely. Let us check one of the more compli-

cated properties, say T10, as an illustration. We must show that every member of the class on the left-hand side of the equation is also a member of the class on the right, and conversely. If $x \in \alpha \cup (\beta \cap \gamma)$, then either $x \in \alpha$ or $x \in \beta \cap \gamma$ or both. If $x \in \alpha$, then certainly $x \in \alpha \cup \beta$ and also $x \in \alpha \cup \gamma$. Hence $x \in (\alpha \cup \beta) \cap (\alpha \cup \gamma)$. Alternatively, if $x \in \beta \cap \gamma$, then $x \in \beta$ and $x \in \gamma$. From the first, $x \in \alpha \cup \beta$, and from the second, $x \in \alpha \cup \gamma$. Hence $x \in (\alpha \cup \beta) \cap (\alpha \cup \gamma)$. We have thus shown that if $x \in \alpha \cup (\beta \cap \gamma)$, then $x \in (\alpha \cup \beta) \cap (\alpha \cup \gamma)$. The converse may be shown in a similar manner. In view of T5 and T7, we shall write $\alpha \cap \beta \cap \gamma$, for $(\alpha \cap \beta) \cap \gamma$, and $\alpha \cup \beta \cup \gamma$ for $(\alpha \cup \beta) \cup \gamma$, etc.

We say that α is included in β, or that α is a subclass of β, (in symbols, $\alpha \subset \beta$) if every member of α is also a member of β, i.e. $x \in \alpha$ always implies that $x \in \beta$. The following propositions are easy to prove:

T21. $\alpha = \beta$ *if and only if* $\alpha \subset \beta$ *and* $\beta \subset \alpha$;
T22. $\alpha \subset \beta$ *if and only if* $\alpha \cap \beta = \alpha$;
T23. $\alpha \subset \beta$ *if and only if* $\alpha \cup \beta = \beta$;
T24. $\alpha \subset \beta$ *if and only if* $\alpha - \beta = 0$;
T25. $\alpha \subset \beta$ *if and only if* $\alpha' \cup \beta = 1$;
T26. $\alpha \subset \alpha$;
T27. $\alpha \cap \beta \subset \alpha \subset \alpha \cup \beta$;
T28. $0 \subset \alpha \subset 1$;
T29. *if* $\alpha \subset 0$, *then* $\alpha = 0$;
T30. *if* $1 \subset \alpha$, *then* $\alpha = 1$;
T31. *if* $\alpha \subset \beta$ *and* $\beta \subset \gamma$, *then* $\alpha \subset \gamma$;
T32. *if* $\alpha \subset \beta$, *then* $\alpha \cap \gamma \subset \beta \cap \gamma$ *and* $\alpha \cup \gamma \subset \beta \cup \gamma$;
T33. *if* $\alpha \subset \beta$, *then* $\beta' \subset \alpha'$;
T34. *if* $\alpha \subset \beta$ *and* $\alpha \subset \gamma$, *then* $\alpha \subset \beta \cap \gamma$;
T35. *if* $\alpha \subset \gamma$ *and* $\beta \subset \gamma$, *then* $\alpha \cup \beta \subset \gamma$.

We have thus shown that if the operations with classes are symbolized in the above fashion, we obtain an algebra similar to our ordinary algebra of numbers. The similarity becomes more striking if we introduce the "exclusive" either–or. Let $\alpha + \beta = (\alpha - \beta) \cup (\beta - \alpha)$, by definition; i.e. $\alpha + \beta$ is the class

of all things which are in one of α and β but not the other. If for the moment we abbreviate "$\alpha \cap \beta$" by "$\alpha\beta$", we obtain the following propositions:

T36. $\alpha\beta = \beta\alpha$; $\alpha(\beta\gamma) = (\alpha\beta)\gamma$;
T37. $\alpha + \beta = \beta + \alpha$; $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$;
T38. $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$;
T39. $\alpha + 0 = \alpha = \alpha 1$;
T40. $\alpha + \alpha = 0$.

It is unnecessary to go back to the original meanings of the symbols in order to prove these statements. We can instead use the properties already stated. Thus

$$\begin{aligned}
\alpha(\beta + \gamma) &= \alpha \cap ((\beta \cap \gamma') \cup (\beta' \cap \gamma)) \\
&= (\alpha \cap \beta \cap \gamma') \cup (\alpha \cap \beta' \cap \gamma) \\
&= \alpha\beta\gamma' \cup \alpha\beta'\gamma, \\
\text{and } \alpha\beta + \alpha\gamma &= ((\alpha \cap \beta) \cap (\alpha \cap \gamma)') \cup ((\alpha \cap \beta)' \cap (\alpha \cap \gamma)) \\
&= ((\alpha \cap \beta) \cap (\alpha' \cup \gamma')) \cup ((\alpha' \cup \beta') \cap (\alpha \cap \gamma)) \\
&= (((\alpha \cap \beta) \cap \alpha') \cup ((\alpha \cap \beta) \cap \gamma')) \\
&\qquad \cup (((\alpha \cap \gamma) \cap \alpha') \cup ((\alpha \cap \gamma) \cap \beta')) \\
&= \alpha\alpha'\beta \cup \alpha\beta\gamma' \cup \alpha\alpha'\gamma \cup \alpha\beta'\gamma \\
&= 0\beta \cup \alpha\beta\gamma' \cup 0\gamma \cup \alpha\beta'\gamma \\
&= 0 \cup \alpha\beta\gamma' \cup 0 \cup \alpha\beta'\gamma \\
&= \alpha\beta\gamma' \cup \alpha\beta'\gamma.
\end{aligned}$$

Here we have used the definition of $\alpha + \beta$, and equations T9, T4, T5, T19, T12, and T13 above.

By virtue of equations T36 to T40 the algebra of classes is what mathematicians call a ring with respect to the operations $\alpha\beta$ and $\alpha + \beta$. Indeed, this ring is a very special one because of T8 and T40, which show that the algebra of this ring is much simpler than our ordinary algebra since there are no exponents or coefficients.

By virtue of T21, T22, T23, T26, T27, T31, T34, and T35, (or alternatively, by T4, T5, T6, T7, T8, and T20) the algebra of classes is also what mathematicians call a lattice. This is a very special type of lattice because of T10 to T13.

We shall not use the knowledge already accumulated concern-

ing rings and lattices in our present work. In more advanced work, however, these points of view are useful.

The algebra of classes is called Boolean algebra after the man (G. Boole, Irish, 1815–1864) who first studied it intensively.

All the formal laws of Boolean algebra can be obtained from one general principle. We must first define the concept of a "Boolean function" step by step. If $f(\alpha) \equiv \gamma$, where γ is a constant class, for all α, then f is a Boolean function. If $f(\alpha) \equiv \alpha$ for all α, then f is a Boolean function, the so-called identity function. If f is a Boolean function, and if $g(\alpha) = (f(\alpha))'$ for all α, then g is a Boolean function. If f and g are Boolean functions, and if $h(\alpha) = f(\alpha) \cup g(\alpha)$ and $k(\alpha) = f(\alpha) \cap g(\alpha)$ for all α, then h and k are Boolean functions. The class of Boolean functions is the smallest class of functions satisfying these conditions, i.e. it is the class of all functions which can be obtained by starting with constants and the identity function, and applying the operations α', $\alpha \cap \beta$, and $\alpha \cup \beta$ a finite number of times. Thus $f(\alpha) \equiv (\gamma \cap \alpha) \cup (\delta \cap \alpha')$, where γ and δ are constant classes, is a Boolean function.

The fundamental theorem of Boolean algebra is

THEOREM 41. *If f is a Boolean function, then*

$$f(\alpha) = (f(1) \cap \alpha) \cup (f(0) \cap \alpha')$$

Proof. If $f(\alpha) \equiv \gamma$, where γ is a constant, then

$$\begin{aligned} (f(1) \cap \alpha) \cup (f(0) \cap \alpha') &= (\gamma \cap \alpha) \cup (\gamma \cap \alpha') \\ &= \gamma \cap (\alpha \cup \alpha') \quad \text{(by T9)} \\ &= \gamma \cap 1 \quad \text{(by T11)} \\ &= \gamma \quad \text{(by T14)} \\ &= f(\alpha). \end{aligned}$$

If $f(\alpha) \equiv \alpha$, then

$$\begin{aligned} (f(1) \cap \alpha) \cup (f(0) \cap \alpha') &= (1 \cap \alpha) \cup (0 \cap \alpha') \\ &= \alpha \cup 0 \quad \text{(by T13, T15)} \\ &= \alpha \quad \text{(by T13)} \\ &= f(\alpha). \end{aligned}$$

Suppose the theorem is true for f. Let $g(\alpha) = (f(\alpha))'$ for all α. Then

$$\begin{aligned}
(\alpha) &= ((f(1) \cap \alpha) \cup (f(0) \cap \alpha'))' \\
&= (f(1) \cap \alpha)' \cap (f(0) \cap \alpha')' && \text{(by T18)} \\
&= (f(1)' \cup \alpha') \cap (f(0)' \cup (\alpha')') && \text{(by T19)} \\
&= (f(1)' \cap f(0)') \cup (f(1)' \cap (\alpha')') \\
&\qquad \cup (\alpha' \cap f(0)') \cup (\alpha' \cap (\alpha')') \\
&\qquad && \text{(by T9, applied twice)} \\
&= (f(1)' \cap f(0)') \cup (f(1)' \cap \alpha) \cup (f(0)' \cap \alpha') \\
&\qquad && \text{(by T16, T12, T13)} \\
&= ((f(1)' \cap f(0)') \cap (\alpha \cup \alpha')) \cup (f(1)' \cap \alpha) \\
&\qquad \cup (f(0)' \cap \alpha') && \text{(by T11)} \\
&= (f(0)' \cap f(1)' \cap \alpha) \cup (f(1)' \cap f(0)' \cap \alpha') \\
&\qquad \cup (f(1)' \cap \alpha) \cup (f(0)' \cap \alpha') && \text{(by T4, T5, T9)} \\
&= (f(1)' \cap \alpha) \cup (f(0)' \cap \alpha') && \text{(by T20).}
\end{aligned}$$

Suppose the theorem is true for f and g, and let $h(\alpha) = f(\alpha) \cup g(\alpha)$ and $k(\alpha) = f(\alpha) \cap g(\alpha)$ for all α. Then

$$\begin{aligned}
h(\alpha) &= (f(1) \cap \alpha) \cup (f(0) \cap \alpha') \cup (g(1) \cap \alpha) \cup (g(0) \cap \alpha') \\
&= ((f(1) \cup g(1)) \cap \alpha) \cup ((f(0) \cup g(0)) \cap \alpha') && \text{(by T9)}
\end{aligned}$$

Also

$$\begin{aligned}
k(\alpha) &= ((f(1) \cap \alpha) \cup (f(0) \cap \alpha')) \cap ((g(1) \cap \alpha) \\
&\qquad \cup (g(0) \cap \alpha')) \\
&= (f(1) \cap g(1) \cap \alpha \cap \alpha) \cup (f(1) \cap g(0) \cap \alpha \cap \alpha') \\
&\qquad \cup (f(0) \cap g(1) \cap \alpha' \cap \alpha) \\
&\qquad \cup (f(0) \cap g(0) \cap \alpha' \cap \alpha') && \text{(by T9, T4, T5)} \\
&= (((f(1) \cap g(1)) \cap \alpha) \cup ((f(0) \cap g(0)) \cap \alpha') \\
&\qquad && \text{(by T8, T12, T13, T15)}
\end{aligned}$$

If f is any Boolean function, then it can be built up in a finite number of steps from constants and the identity function by means of the operations α', $\alpha \cap \beta$, and $\alpha \cup \beta$. Therefore, by combining these results, we immediately obtain the theorem.

This theorem shows that in order to prove that two Boolean functions, f and g, are equal for all α, it is sufficient to prove that $f(0) = g(0)$ and $f(1) = g(1)$.

All these considerations can be extended to Boolean functions of several variables. Thus if f is a Boolean function of two variables, then

$$f(\alpha, \beta) = (f(1, 1)\alpha\beta) \cup (f(1, 0)\alpha\beta') \cup (f(0, 1)\alpha'\beta) \cup (f(0, 0)\alpha'\beta').$$

As a corollary we obtain

$$f(\alpha \cup \beta) \cup f(\alpha \cap \beta) = f(\alpha) \cup f(\beta),$$

if f is any Boolean function. For let $g(\alpha, \beta) = f(\alpha \cup \beta) \cup f(\alpha \cap \beta)$, $h(\alpha, \beta) = f(\alpha) \cup f(\beta)$. Then $g(1, 1) = h(1, 1)$, $g(1, 0) = h(1, 0)$, etc. Therefore, $g(\alpha, \beta) = h(\alpha, \beta)$ for all α and β.

EXERCISES

Ex. 1. Verify T4–T40.

Ex. 2. (a). Prove T8 from T27, T34, and T21.
(b). Prove T11 from T26 and T25.
(c). Prove that $\alpha'' \subset \alpha$ from T11, T6, and T25.
(d). Prove T35 from T34, T33, T18, and T16.

Ex. 3. Show that if f is any Boolean function of one variable, then
(a). $f(\alpha) = (f(1) \cup \alpha') \cap (f(0) \cup \alpha)$.
(b). $f(f(0)) = f(0) \cap f(1) \subset f(\alpha) \subset f(0) \cup f(1) = f(f(1))$.
(c). $f(\alpha) = \gamma + \delta\alpha$, where γ and δ are constants.
(d). If $f(0) \cap f(1) \subset \eta \subset f(0) \cup f(1)$, then the equation $f(\alpha) = \eta$ has a solution. Find all solutions.
(e). If the equation $f(\xi) = \eta$ has a unique solution for one value of η, then it has a unique solution, namely $f(\eta)$, for all values of η.
(f). If $f(\xi_1) \subset f(\xi_2)$ whenever $\alpha \subset \xi_1 \subset \xi_2 \subset \beta$, then $\beta \cap f(0) \subset \alpha \cup f(1)$, and conversely.
(g). If $\xi \subset \eta$, then $f(f(\xi)) \subset f(f(\eta))$.
(h). If $\alpha \subset \beta$ and $f(\alpha) \subset f(\beta)$, then $f(\xi_1) \subset f(\xi_2)$ whenever $\alpha \subset \xi_1 \subset \xi_2 \subset \beta$.