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Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science

Volume III: Foundations and Philosophy of Statistical Theories in the Physical Sciences

W. L. Harper & C. A. Hooker
(editors)

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FOUNDATIONS OF PROBABILITY THEORY, STATISTICAL INFERENCE, AND STATISTICAL THEORIES OF SCIENCE

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VOLUME III

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Foundations and Philosophy of Statistical Inference

GENERAL INTRODUCTION

In May of 1973 we organized an international research colloquium on foundations of probability, statistics, and statistical theories of science at the University of Western Ontario.

During the past four decades there have been striking formal advances in our understanding of logic, semantics and algebraic structure in probabilistic and statistical theories. These advances, which include the development of the relations between semantics and metamathematics, between logics and algebras and the algebraic-geometrical foundations of statistical theories (especially in the sciences), have led to striking new insights into the formal and conceptual structure of probability and statistical theory and their scientific applications in the form of scientific theory.

The foundations of statistics are in a state of profound conflict. Fisher's objections to some aspects of Neyman-Pearson statistics have long been well known. More recently the emergence of Bayesian statistics as a radical alternative to standard views has made the conflict especially acute. In recent years the response of many practising statisticians to the conflict has been an eclectic approach to statistical inference. Many good statisticians have developed a kind of wisdom which enables them to know which problems are most appropriately handled by each of the methods available. The search for principles which would explain why each of the methods works where it does and fails where it does offers a fruitful approach to the controversy over foundations. The colloquium first aimed both at a conceptually exciting clarification and enrichment of our notion of a probability theory and at removing the cloud hanging over many of the central methods of statistical testing now in constant use within the social and natural sciences.

The second aim of the colloquium was that of exploiting the same formal developments in the structure of probability and statistical theories for an understanding of what it is to have a statistical theory of nature, or of a sentient population. A previous colloquium in this series has already examined thoroughly the recent development of the analysis of quantum mechanics in terms of its logico-algebraic structure and brought out many of the sharp and powerful insights into the basic physical significance of this theory which that formal approach provides. It was our aim in this colloquium to extend the scope of that inquiry yet further afield in an effort to understand, not just one particular idiosyncratic theory, but what it is in general we are doing when we lay down a formal statistical theory of a system (be it physical or social).

Our aim was to provide a workshop context in which the papers presented could benefit from the informed criticism of conference participants. Most of the papers that appear here have been considerably rewritten since their original presentation. We have also included comments by other participants and replies wherever possible. One of the main reasons we have taken so long to get these proceedings to press has been the time required to obtain final versions of comments and replies. We feel that the result has been worth the wait.

When the revised papers came in there was too much material to include in a single volume or even in two volumes. We have, therefore, broken the proceedings down into three volumes. Three basic problem areas emerged in the course of the conference and our three volumes correspond. Volume I deals with problems in the foundations of probability theory; Volume II is devoted to foundations of statistical inference, and Volume III is devoted to statistical theories in the physical sciences. There is considerable overlap in these areas so that in some cases a paper in one volume might just as well have gone into another.

INTRODUCTION TO VOLUME III

One of the major influences on recent developments in foundations research in probability theory has been the striking role of probability theory in twentieth century scientific theories. Relativity theory aside, twentieth century theoretical physics research has been dominated by the exploration of statistical mechanics and quantum theory, and by the search for an understanding of the relation between them. Quantum theoretic use of probability theory has raised increasingly profound questions about the mathematical and physical conception of probability. The papers in this volume are almost entirely devoted to the exploration of these issues – but with a significant twist: most of them belong to an emerging tradition of foundations research in physics, of great power and potential, which takes a highly abstract approach to physical theory. Concerning this tradition one of us (CAH) wrote, in the introduction of a related work [2], as follows:

> The twentieth century has witnessed a striking transformation in the understanding of the theories of mathematical physics. There has emerged clearly the idea that physical theories are significantly characterized by their abstract mathematical structure. This is in opposition to the traditional opinion that one should look to the specific applications of a theory in order to understand it. One might with reason now espouse the view that to understand the deeper character of a theory one must know its abstract structure and understand the significance of that structure, while to understand how a theory might be modified in light of its experimental inadequacies one must be intimately acquainted with how it is applied.
>
> Quantum theory itself has gone through a development this century which illustrates strikingly the shifting perspective. From a collection of intuitive physical maneuvers under Bohr, through a formative stage in which the mathematical framework was bifurcated (between Schrödinger and Heisenberg) to an elegant culmination in von Neumann's Hilbert space formulation the elementary theory moved, flanked even at the later stage by the ill-understood formalisms for the relativistic version and for the field-theoretic alternative; after that we have a gradual, but constant, elaboration of all these quantal theories as abstract mathematical structures (their point of departure being von Neumann's formalism) until at the present time theoretical work is heavily preoccupied with the manipulation of purely abstract structures.

The papers by Bub, Demopoulos, Finch, Finkelstein, Gudder, Mittelstaedt and Randall and Foulis all belong, in their several distinct ways, to this tradition. (If the reader will consult their other publications, in the light of [1] and [2], he will discern an interlocking pattern of development in this abstract approach to physical theory.) There is emerging from this work a new and deeper understanding of the role of probability structures in theoretical physics.

The papers by Chernoff and Marsden and by Tisza, too, belong to an abstract structural approach, though not of a purely or strongly logico-algebraic sort. Bunge offers a clarification of the logico-semantic framework for probability in physics (cf. Tisza). Lubkin remarks on interesting applications of a generalized quantum approach to other subject matters.

W. L. HARPER and C. A. HOOKER

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[1] *Contemporary Research in the Foundations and Philosophy of Quantum Theory* (ed. by C. A. Hooker), Reidel, Dordrecht, 1974.

[2] *The Logico-Algebraic Approach to Quantum Theory* (ed. by C. A. Hooker), Vol. I, 1975, Vol. II, 1976.

CONTENTS

JEFFREY BUB

THE STATISTICS OF NON-BOOLEAN EVENT STRUCTURES

Quantum mechanics incorporates an algorithm for assigning probabilities to ranges of values of the physical magnitudes of a mechanical system:

$$p_W(a \in S) = \mathrm{Tr}(WP_A(S))$$

where W represents a statistical state, and $P_A(S)$ is the projection operator onto the subspace in the Hilbert space of the system associated with the range S of the magnitude A. (I denote values of A by a.) The statistical states (represented by the statistical operators in Hilbert space) generate all possible (generalized) probability measures on the partial Boolean algebra of subspaces of the Hilbert space.[1] Joint probabilities

$$P_W(a_1 \in S_1 \,\&\, a_2 \in S_2 \,\&\ldots\, \&\, a_n \in S_n) = \\ = \mathrm{Tr}(WP_{A_1}(S_1)\,P_{A_2}(S_2)\ldots P_{A_n}(S_n))$$

are defined only for compatible[2] magnitudes $A_1, A_2, \ldots A_n$, and there are no dispersion-free statistical states.

The problem of hidden variables concerns the possibility of representing the statistical states of a quantum mechanical system by measures on a classical probability space, in such a way that the algebraic structure of the magnitudes is preserved.[3] This is the problem of imbedding the partial algebra of magnitudes in a commutative algebra or, equivalently, the problem of imbedding the partial Boolean algebra of idempotent magnitudes (properties, propositions) in a Boolean algebra. The imbedding turns out to be impossible; there are no 2-valued homomorphisms on the partial Boolean algebra of idempotents of a quantum mechanical system, except in the case of a system associated with a 2-dimensional Hilbert space.[4]

Thus, the transition from classical to quantum mechanics involves the generalization of the Boolean propositional or event structures of classical mechanics[5] to a particular class of non-Boolean structures. This may be understood as a generalization of the classical notion of validity: the class of models over which validity is defined is extended to include partial Boolean algebras which are not imbeddable in Boolean algebras.[6]

Harper and Hooker (eds.), Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science, Vol. III, 1–16.

In a Boolean algebra $\mathscr{B}$, there is a 1-1 correspondence between atoms, ultrafilters, and 2-valued homomorphisms, essentially because an ultrafilter Φ in $\mathscr{B}$ contains a or a', but not both, for every $a \in \mathscr{B}$.[7] In a partial Boolean algebra $\mathscr{A}$ that is not imbeddable in a Boolean algebra, this correspondence no longer holds. The partial Boolean algebra may be regarded as a partially ordered system, so the notion of a filter (and hence an ultrafilter as a maximal filter) is still well-defined. But it is no longer the case that if Φ is an ultrafilter, then for each $a \in \mathscr{A}$ either $a \in \Phi$ or $a' \in \Phi$, and hence ultrafilters do not define 2-valued homomorphisms on $\mathscr{A}$.[8] An atom in $\mathscr{A}$ will correspond to an ultrafilter but not to a prime filter, and hence will not define a 2-valued homomorphism on $\mathscr{A}$.

A measure on a classical probability space $(X, \mathscr{F}, \mu)$ may be interpreted as a measure over ultrafilters or atoms in a Boolean algebra $\mathscr{B}$, the points $x \in X$ corresponding to ultrafilters in $\mathscr{B}$ and the singleton subsets $\{x\}$ in $\mathscr{F}$ corresponding to atoms in $\mathscr{B}$. (Under the Stone isomorphism,[9] every element in a Boolean algebra is mapped onto the set of ultrafilters containing the element.) Thus, the probability of an event a[10] may be understood as the measure of the set of ultrafilters containing a, or the measure of the set of atomic events that can occur together with the event a:

$$p(a) = \mu(\Phi_a)$$

The conditional probability of a given b, $p(a \mid b)$, is the measure of the set of ultrafilters containing a in the set of ultrafilters containing b, with respect to a renormalized measure assigning probability 1 to the set Φ_b:

$$p(a \mid b) = \frac{\mu(\phi_a \cap \phi_b)}{\mu(\phi_b)} = \frac{p(a \wedge b)}{p(b)}$$

Loosely: We 'count' the number of atomic events that can occur together with the event b, in the set of atomic events that can occur together with the event a. (Notice that if b is an atom, the conditional probability is a 2-valued measure.)

The statistical states of quantum mechanics define probability measures in the classical sense on each maximal Boolean subalgebra of the partial Boolean algebra of propositions of a quantum mechanical system. Consider a system associated with a 3-dimensional Hilbert space $\mathscr{H}_3$. Let A and B be two incompatible (non-degenerate)[11] magnitudes with eigenvalues a_1, a_2, a_3 and b_1, b_2, b_3, respectively. The corresponding eigenvectors are $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$. I shall also denote the atoms (atomic propositions or events) in the maximal Boolean subalgebras $\mathscr{B}_A$ and $\mathscr{B}_B$ of $\mathscr{A}_3$ by a_i and b_j, i.e. I shall use the same symbols to denote properties of the systems represented by these values of the magnitudes.

The statistical state with the vector α_1 assigns probabilities

$$p_{\alpha_1}(a_1) = 1, \quad p_{\alpha_1}(a_2) = 0, \quad p_{\alpha_1}(a_3) = 0$$

to the atomic propositions in $\mathscr{B}_A$, and probabilities

$$p_{\alpha_1}(b_1) = |(\beta_1, \alpha_1)|^2, \quad p_{\alpha_1}(b_2) = |(\beta_2, \alpha_1)|^2,$$
$$p_{\alpha_1}(b_3) = |(\beta_3, \alpha_1)|^2$$

to the atomic propositions in $\mathscr{B}_B$. How are these probabilities to be understood? The problem at issue is this: Suppose a system S has the property a_1. The statistical algorithm of quantum mechanics assigns non-zero probabilities to properties incompatible with a_1, for example $p_{\alpha_1}(b_1) = |(\beta_1, \alpha_1)|^2$. Now, the probability assigned to b_1 by the statistical state p_{α_1} (the projection operator onto the 1-dimensional subspace spanned by the vector α_1) cannot be interpreted as the relative measure of the set of ultrafilters containing b_1 in the set of ultrafilters containing a_1 because, firstly, a_1 and b_1 are atoms in $\mathscr{A}_3$ and, secondly, a_1 and b_1 cannot be represented as non-atomic properties in a Boolean algebra because no Boolean imbedding of $\mathscr{A}_3$ is possible. Thus, since there are no 2-valued homomorphisms on $\mathscr{A}_3$, the probability $p_{\alpha_1}(b_1)$ cannot be interpreted as the conditional probability, $p(b_1 \mid a_1)$, that the proposition b_1 is true (or the corresponding event obtains) given that the proposition a_1 is true, i.e. the probability that the value of the magnitude B is b_1 given that the value of the magnitude A is a_1. What do these probabilities mean?

Usually, the problem of interpreting the quantum statistics is posed rather differently. It is pointed out that the statistics defined by p_{α_1}, for the magnitude B cannot be understood in terms of a statistical ensemble constituted of systems in quantum states[12] $\beta_1, \beta_2, \beta_3$ with weights $|(\beta_1, \alpha_1)|^2, |(\beta_2, \alpha_1)|^2, |(\beta_3, \alpha_1)|^2$. Such an ensemble is represented by the statistical operator

$$W = \sum_{i=1}^{3} |\beta_i, \alpha_1)|^2 \, P_{\beta_i}$$

which yields the same statistics as P_{α_1} only for magnitudes compatible with B. Thus, the problem of interpreting the quantum statistical relations between incompatible magnitudes is presented as the problem of making sense of the statistics of *pure ensembles*, insofar as these ensembles are not reducible to *mixtures*: probability distributions of systems in quantum states represented by Hilbert space vectors.

Those interpretations which follow the Copenhagen interpretation of Bohr and Heisenberg propose that a (micro-) system, at any one time, is characterized by a set of properties which form a Boolean algebra – a maximal Boolean subalgebra in the partial Boolean algebra of quantum mechanical properties.[13] The appropriate Boolean subalgebra is related to the experimental conditions defined by macroscopic measuring instruments. In effect, this amounts to saying that a system is always represented by some mixture of quantum states (in the limiting case, by a mixture with 0, 1 weights), the constituents of the mixture being determined by the experimental conditions. If the experimental conditions are such as to determine the maximal Boolean subalgebra $\mathscr{B}_A$ associated with the (non-degenerate) magnitude A, then the system is actually in one of the states α_1, α_2, α_3, and hence represented by a statistical operator of the form

$$\sum_{i=1}^{3} w_i P_{\alpha_i}.$$

If the state is known, $w_i = 1$ or 0. The probabilities assigned to the 'complementary' propositions in $\mathscr{B}_B$ by a pure statistical operator P_{α_i}, representing a statistical ensemble of systems all in the state α_i, are to be understood as the probabilities of finding particular B-values, if the experimental conditions are altered so as to determine the maximal Boolean subalgebra $\mathscr{B}_B$, given that the system is in the state α_i. The quantum mechanical description of a physical system, in terms of a partial Boolean algebra of properties, allows the consideration of all possible experimental conditions, but the application of this description to a particular system is always with respect to the experimental conditions obtaining for the system, which specify a particular maximal Boolean subalgebra of properties.

The Copenhagen interpretation leads to an insoluble measurement problem, if the experimental conditions for a system $S^{(1)}$ are assumed to be determined (in principle, at least) by a physical interaction between $S^{(1)}$ and a second system $S^{(2)}$ (which, even if macroscopic, ought to be reducible to a complex of interacting microsystems).[14] Suppose that a particular maximal Boolean subalgebra $\mathscr{B}_{A^{(1)}}$[15] is selected for $S^{(1)}$ by an interaction with $S^{(2)}$ governed by the quantum mechanical equation of motion. Suppose, further, that $S^{(2)}$ is a measuring instrument suitable for measuring the $S^{(1)}$-magnitude $A^{(1)}$, so that correlations are established during the interaction between the possible values of $A^{(1)}$ and the possible values of an $S^{(2)}$-magnitude $A^{(2)}$ (in the sense that the probability assigned by the statistical operator of the composite system $S^{(1)}+S^{(2)}$ to the pair of values $a_i^{(1)}$, $a_j^{(2)}$ $(i \neq j)$ is zero). If the initial quantum state of $S^{(1)}$ is represented by the vector

$$\psi = \sum_i (\alpha_i^{(1)}, \psi)\, \alpha_i^{(1)},$$

what is apparently required is an interaction which results in the representation

$$W = \sum_i w_i P_{\alpha_i^{(1)} \otimes \alpha_i^{(2)}},$$

with $w_i = |(\alpha_i^{(1)}, \psi)|^2$, for the statistical state of the composite system $S^{(1)}+S^{(2)}$.[16] Since the statistics defined by W for $S^{(1)}$ is given by the operator[17]

$$W^{(1)} = \sum_i w_i P_{\alpha_i^{(1)}},$$

the quantum state of the system $S^{(1)}$ may be regarded as belonging to the set $\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)}$, and the probabilities w_i as a measure of our ignorance of the actual state.

Now, of course, there can be no quantum mechanical interaction which results in the transition

$$P_{\Psi_0} \to W$$

where P_{Ψ_0} is the statistical operator of the initial pure ensemble of composite systems all in the quantum state Ψ_0.[18] For the evolution of a quantum mechanical system is governed by Schrödinger's equation of motion, represented by a *unitary* transformation of the Hilbert space vector defining the quantum state of the system, and the transition $P_{\Psi_0} \to W$, from the pure ensemble represented by P_{Ψ_0} to the mixture represented by W, is *non-unitary*.

Attempted solutions to this problem exploit the similarity between W

and the statistical operator P_Ψ, with

$$\psi = \sum_i (\alpha_i^{(1)}, \psi)\, \alpha_i^{(1)} \otimes \alpha_i^{(2)}.$$

The transition $P_{\Psi_0} \to P_\Psi$ is unitary, and the statistics defined by P_Ψ for $S^{(1)}$ is also given by the operator

$$W^{(1)} = \sum_i w_i P_{\alpha_i(1)}.$$

But W may be taken as representing a statistical ensemble of systems in quantum states $\alpha_i^{(1)} \otimes \alpha_i^{(2)}$ with weights w_i, which is manifestly different from the pure ensemble represented by P_Ψ and constituted of systems each in the state Ψ. In fact, P_Ψ and W define the same statistics for $S^{(1)}$-magnitudes, $S^{(2)}$-magnitudes, and for $S^{(1)}+S^{(2)}$-magnitudes compatible with the magnitude $A^{(1)}+A^{(2)}$ (whose eigenvectors are $\alpha_i^{(1)} \otimes \alpha_j^{(2)}$), but differ for general $S^{(1)}+S^{(2)}$-magnitudes.[19]

Actually, these considerations, which are usually laboured in discussions on the measurement problem, are largely irrelevant. The objections to the transition $P_{\Psi_0} \to P_\Psi$ as a process which selects a maximal Boolean subalgebra of propositions for the measured system $S^{(1)}$ apply with equal force to the transition $P_{\Psi_0} \to W$. Clearly, P_Ψ does not yield a representation for $W^{(1)}$ as a unique mixture of orthogonal pure states, represented by the eigenvectors of the magnitude measured. For we may have

$$\begin{aligned} \Psi &= \sum_i (\alpha_1^{(1)}, \psi)\, \alpha_i^{(1)} \otimes \alpha_i^{(2)} \\ &= \sum_j (\beta_j^{(1)} \otimes \beta_j^{(2)}, \Psi)\, \beta_j^{(1)} \otimes \beta_j^{(2)} \end{aligned}$$

where $\beta_j^{(1)}$, $\beta_j^{(2)}$ ($j = 1, 2, 3$) are the eigenvectors of magnitudes $B^{(1)}$, $B^{(2)}$ incompatible with $A^{(1)}$, $A^{(2)}$, respectively, so that the statistics defined by P_Ψ for $S^{(1)}$ is given by the operator

$$W^{(1)} = \sum_i w_i P_{\alpha_i(1)} = \sum_j w'_j P_{\beta_j(1)}$$

with $w_i = |(\alpha_i^{(1)}, \psi)|^2$, $w'_j = |(\beta_j^{(1)} \otimes \beta_j^{(2)}, \Psi)|^2$.[20] But since the representation of a statistical operator as a weighted sum of pure statistical operators is in general not unique,[21] even assuming some physical grounds for restricting the representation to statistical operators associated with orthogonal vectors in Hilbert space, the mixture defined by $W =$