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V.I. Arnold

Mathematical Methods of Classical Mechanics

Second Edition

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Translated by K. Vogtmann
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Preface

Many different mathematical methods and concepts are used in classical mechanics: differential equations and phase flows, smooth mappings and manifolds, Lie groups and Lie algebras, symplectic geometry and ergodic theory. Many modern mathematical theories arose from problems in mechanics and only later acquired that axiomatic-abstract form which makes them so hard to study.

In this book we construct the mathematical apparatus of classical mechanics from the very beginning; thus, the reader is not assumed to have any previous knowledge beyond standard courses in analysis (differential and integral calculus, differential equations), geometry (vector spaces, vectors) and linear algebra (linear operators, quadratic forms).

With the help of this apparatus, we examine all the basic problems in dynamics, including the theory of oscillations, the theory of rigid body motion, and the hamiltonian formalism. The author has tried to show the geometric, qualitative aspect of phenomena. In this respect the book is closer to courses in theoretical mechanics for theoretical physicists than to traditional courses in theoretical mechanics as taught by mathematicians.

A considerable part of the book is devoted to variational principles and analytical dynamics. Characterizing analytical dynamics in his "Lectures on the development of mathematics in the nineteenth century," F. Klein wrote that ". . . a physicist, for his problems, can extract from these theories only very little, and an engineer nothing." The development of the sciences in the following years decisively disproved this remark. Hamiltonian formalism lay at the basis of quantum mechanics and has become one of the most often used tools in the mathematical arsenal of physics. After the significance of symplectic structures and Huygens' principle for all sorts of optimization problems was realized, Hamilton's equations began to be used constantly in engineering calculations. On the other hand, the contemporary development of celestial mechanics, connected with the requirements of space exploration, created new interest in the methods and problems of analytical dynamics.

The connections between classical mechanics and other areas of mathematics and physics are many and varied. The appendices to this book are devoted to a few of these connections. The apparatus of classical mechanics is applied to: the foundations of riemannian geometry, the dynamics of an ideal fluid, Kolmogorov's theory of perturbations of conditionally periodic motion, short-wave asymptotics for equations of mathematical physics, and the classification of caustics in geometrical optics.

These appendices are intended for the interested reader and are not part of the required general course. Some of them could constitute the basis of special courses (for example, on asymptotic methods in the theory of non-linear oscillations or on quasi-classical asymptotics). The appendices also contain some information of a reference nature (for example, a list of normal forms of quadratic hamiltonians). While in the basic chapters of the book the author has tried to develop all the proofs as explicitly as possible, avoiding references to other sources, the appendices consist on the whole of summaries of results, the proofs of which are to be found in the cited literature.

The basis for the book was a year-and-a-half-long required course in classical mechanics, taught by the author to third- and fourth-year mathematics students at the mathematics-mechanics faculty of Moscow State University in 1966–1968.

The author is grateful to I. G. Petrovsky, who insisted that these lectures be delivered, written up, and published. In preparing these lectures for publication, the author found very helpful the lecture notes of L. A. Bunimovich, L. D. Vaingortin, V. L. Novikov, and especially, the mimeographed edition (Moscow State University, 1968) organized by N. N. Kolesnikov. The author thanks them, and also all the students and colleagues who communicated their remarks on the mimeographed text; many of these remarks were used in the preparation of the present edition. The author is grateful to M. A. Leontovich, for suggesting the treatment of connections by means of a limit process, and also to I. I. Vorovich and V. I. Yudovich for their detailed review of the manuscript.

V. ARNOLD

The translators would like to thank Dr. R. Barrar for his help in reading the proofs. We would also like to thank many readers, especially Ted Courant, for spotting errors in the first two printings.

Berkeley, 1981

K. VOGTMANN
A. WEINSTEIN

Preface to the second edition

The main part of this book was written thirty years ago. The ideas and methods of symplectic geometry, developed in this book, have now found many applications in mathematical physics and in other domains of applied mathematics, as well as in pure mathematics itself. Especially, the short-wave asymptotical expansions theory has reached a very sophisticated level, with many important applications to optics, wave theory, acoustics, spectroscopy, and even chemistry; this development was parallel to the development of the theories of Lagrange and Legendre singularities, that is, of singularities of caustics and of wave fronts, of their topology and their perestroikas (in Russian metamorphoses were always called "perestroikas," as in "Morse perestroika" for the English "Morse surgery"; now that the word perestroika has become international, we may preserve the Russian term in translation and are not obliged to substitute "metamorphoses" for "perestroikas" when speaking of wave fronts, caustics, and so on).

Integrable hamiltonian systems have been discovered unexpectedly in many classical problems of mathematical physics, and their study has led to new results in both physics and mathematics, for instance, in algebraic geometry.

Symplectic topology has become one of the most promising and active brances of "global analysis." An important generalization of the Poincaré "geometric theorem" (see Appendix 9) was proved by C. Conley and E. Zehnder in 1983. A sequence of works (by M. Chaperon, A. Weinstein, J.-C. Sikorav, M. Gromov, Ja. M. Eliashberg, Ju. Tchekanov, A. Floer, C. Viterbo, H. Hofer, and others) marks important progress in this very living domain. One may hope that this progress will lead to the proof of many known conjectures in symplectic and contact topology, and to the discovery of new results in this new domain of mathematics, emerging from the problems of mechanics and optics.

The present edition includes three new appendices. They represent the modern development of the theory of ray systems (the theory of singularity and of perestroikas of caustics and of wave fronts, related to the theory of Coxeter reflection groups), the theory of integrable systems (the geometric theory of elliptic coordinates, adapted to the infinite-dimensional Hilbert space generalization), and the theory of Poisson structures (which is a generalization of the theory of symplectic structures, including degenerate Poisson brackets).

A more detailed account of the present state of perturbation theory may be found in the book, *Mathematical Aspects of Classical and Celestial Mechanics* by V. I. Arnold, V. V. Kozlov, and A. I. Neistadt, Encyclopaedia of Math. Sci., Vol. 3 (Springer, 1986); Volume 4 of this series (1988) contains a survey "Symplectic geometry" by V. I. Arnold and A. B. Givental', an article by A. A. Kirillov on geometric quantization, and a survey of the modern theory of integrable systems by S. P. Novikov, I. M. Krichevez, and B. A. Doubrovin.

For more details on the geometry of ray systems, see the book *Singularities of Differentiable Mappings* by V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko (Vol. 1, Birkhäuser 1985; vol. 2, Birkhäuser, 1988). *Catastrophe Theory* by V. I. Arnold (Springer, 1986) (second edition) contains a long annotated bibliography.

Surveys on symplectic and contact geometry and on their applications may be found in the Bourbaki seminar (D. Bennequin, "Caustiques mistiques", February, 1986) and in a series of articles (V. I. Arnold, First steps of symplectic topology, Russian Math. Surveys, 41 (1986); Singularities of ray systems, Russian Math. Surveys, 38 (1983); Singularities in variational calculus, Modern Problems of Math., VINITI, 22 (1983) (translated in J. Soviet Math.); and O. P. Shcherbak, Wave fronts and reflection groups, Russian Math. Surveys, 43 (1988)).

Volumes 22 (1983) and 33 (1988) of the VINITI series, "Sovremennye problemy mathematiki. Noveishie dostijenia," contain a dozen articles on the applications of symplectic and contact geometry and singularity theory to mathematics and physics.

Bifurcation theory (both for hamiltonian and for more general systems) is discussed in the textbook *Geometrical Methods of the Theory of Ordinary Differential Equations* (Springer, 1988) (this new edition is more complete than the preceding one). The survey "Bifurcation theory and its applications in mathematics and mechanics" (XVIIth International Congress of Theoretical and Applied Mechanics in Grenoble, August, 1988) also contains new information, as does Volume 5 of the Encyclopaedia of Math. Sci. (Springer, 1989), containing the survey "Bifurcation theory" by V. I. Arnold, V. S. Afraimovich, Ju. S. Aljashenko, and L. P. Shilnikov. Volume 2 of this series, edited by D. V. Anosov and Ja. G. Sinai, is devoted to the ergodic theory of dynamical systems including those of mechanics.

The new discoveries in all these theories have potentially extremely wide applications, but since these results were discovered rather recently, they are discussed only in the specialized editions, and applications are impeded by the difficulty of the mathematical exposition for nonmathematicians. I hope that the present book will help to master these new theories not only to mathematicians, but also to all those readers who use the theory of dynamical systems, symplectic geometry, and the calculus of variations—in physics, mechanics, control theory, and so on.

December 1988 V. I. Arnold

Translator's preface to the second edition

This edition contains three new appendices, originally written for inclusion in a German edition. They describe work by the author and his co-workers on Poisson structures, elliptic coordinates with applications to integrable systems, and singularities of ray systems. In addition, numerous corrections to errors found by the author, the translators, and readers have been incorporated into the text.

Contents

PART I
NEWTONIAN MECHANICS

Newtonian mechanics studies the motion of a system of point masses in three-dimensional euclidean space. The basic ideas and theorems of newtonian mechanics (even when formulated in terms of three-dimensional cartesian coordinates) are invariant with respect to the six-dimensional[1] group of euclidean motions of this space.

A newtonian potential mechanical system is specified by the masses of the points and by the potential energy. The motions of space which leave the potential energy invariant correspond to laws of conservation.

Newton's equations allow one to solve completely a series of important problems in mechanics, including the problem of motion in a central force field.

[1] And also with respect to the larger group of galilean transformations of space-time.

Experimental facts 1

In this chapter we write down the basic experimental facts which lie at the foundation of mechanics: Galileo's principle of relativity and Newton's differential equation. We examine constraints on the equation of motion imposed by the relativity principle, and we mention some simple examples.

1 The principles of relativity and determinacy

In this paragraph we introduce and discuss the notion of an inertial coordinate system. The mathematical statements of this paragraph are formulated exactly in the next paragraph.

A series of experimental facts is at the basis of classical mechanics.[2] We list some of them.

A *Space and time*

Our space is three-dimensional and euclidean, and time is one-dimensional.

B *Galileo's principle of relativity*

There exist coordinate systems (called inertial) possessing the following two properties:

1. All the laws of nature at all moments of time are the same in all inertial coordinate systems.
2. All coordinate systems in uniform rectilinear motion with respect to an inertial one are themselves inertial.

[2] All these "experimental facts" are only approximately true and can be refuted by more exact experiments. In order to avoid cumbersome expressions, we will not specify this from now on and we will speak of our mathematical models as if they exactly described physical phenomena.

In other words, if a coordinate system attached to the earth is inertial, then an experimenter on a train which is moving uniformly in a straight line with respect to the earth cannot detect the motion of the train by experiments conducted entirely inside his car.

In reality, the coordinate system associated with the earth is only approximately inertial. Coordinate systems associated with the sun, the stars, etc. are more nearly inertial.

C *Newton's principle of determinacy*

The initial state of a mechanical system (the totality of positions and velocities of its points at some moment of time) uniquely determines all of its motion.

It is hard to doubt this fact, since we learn it very early. One can imagine a world in which to determine the future of a system one must also know the acceleration at the initial moment, but experience shows us that our world is not like this.

2 The galilean group and Newton's equations

In this paragraph we define and investigate the galilean group of space-time transformations. Then we consider Newton's equation and the simplest constraints imposed on its right-hand side by the property of invariance with respect to galilean transformations.[3]

A *Notation*

We denote the set of all real numbers by $\mathbb{R}$. We denote by $\mathbb{R}^n$ an n-dimensional real vector space.

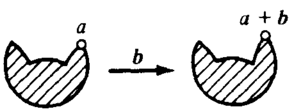

Figure 1 Parallel displacement

Affine n-dimensional space A^n is distinguished from $\mathbb{R}^n$ in that there is "no fixed origin." The group $\mathbb{R}^n$ acts on A^n as the *group of parallel displacements* (Figure 1):

$$a \to a + \mathbf{b}, \qquad a \in A^n, \mathbf{b} \in \mathbb{R}^n, a + \mathbf{b} \in A^n.$$

[Thus the sum of two points of A^n is not defined, but their difference is defined and is a vector in $\mathbb{R}^n$.]

[3] The reader who has no need for the mathematical formulation of the assertions of Section 1 can omit this section.