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PREFACE

In 1981 a Japanese-Polish cooperation in the subject "Numerical Methods of Optimization and Game Theory" was established. It was sponsored and supported respectively by the Japanese Society for Promotion of Science and by the Polish Academy of Sciences.

This cooperation involved scientists from Hiroshima and Osaka Universities on the Japanese side and the Systems Research Institute of the Polish Academy of Sciences on the Polish side.

The cooperation resulted, among others, in a number of joint papers and a book "Constructive Aspects of Optimization", Państwowe Wydawnictwo Naukowe (Polish Scientific Publishers), Warszawa-Łódź, 1985.

Formally the cooperation ended in 1983 but it has been continued on informal basis and the present book is its result.

Among the authors contributing to the book Professor Koichi Mizukami and Mr. Naofumi Iwata are with the Department of the Information and Behavioral Sciences of the Hiroshima University, Professor Yoshiyuki Sakawa and Dr. Yuji Shindo are with the Department of Control Engineering of the Osaka University, Professor Constantin Vârsan is with the Department of Mathematics, National Institute for Scientific and Technical Creation, Bucharest, Romania, while all remaining contributors are from the Systems Research Institute of the Polish Academy of Sciences.

K. Malanowski

K. Mizukami

INTRODUCTION

This book consists of ten independent and self-contained chapters written by different authors. Most of the presented results belong to the authors themselves. They are either published here for the first time or having been partially published elsewhere they are presented here in a different form. The proofs of the results are sometime omitted, but the detailed references are provided.

The main part of the presented material (Chapter 1 through 6) is devoted to modelling and optimization of distributed parameter systems. Thus, among them the three first chapters concern theoretical aspects of optimization, while in the next three some numerical problems are presented.

The material presented in Chapter 7 is very close to that in the previous part, namely there is proposed an iterative algorithm of solving some optimal control problems for systems described by ordinary differential equations.

Chapters 8 and 9 are devoted to some game-theoretical problems. Finally, Chapter 10 concerns calculation of the so called surrogate constraints in mathematical programming problems.

A short outline of the results presented in all chapters is given below.

Chapters 1 and 2 concern sensitivity analysis of solutions to optimal control problems for distributed parameter systems.

More precisely, the dependence of solutions on a parameter, which enters the data of convex optimal control problems subject to inequality-type constraints is investigated. It is shown that in the considered cases the solutions to the optimization problems, as functions of the parameter, are directionally (conically) differentiable and the respective right-derivatives can be found effectively as the solutions to auxiliary quadratic optimal control problems. Sufficient conditions under which these functions are Gâteaux differentiable are obtained.

In this analysis the main difficulty is created by the presence of inequality type constraints. In Chapters 1 and 2 two different methods of coping with this difficulty are presented.

In Chapter 1 the results of the directional differentiability of the mapping of projection onto a closed and convex set in a Hilbert space are exploited to obtain differentiability of solutions for some state and control constrained optimal control problems with linear constraints.

In Chapter 2 the Lagrange formalism for optimal control problems subject to convex pointwise constraints is used. This approach allows to analyse the differential properties of both the solutions and the associated Lagrange multipliers, it requires however, Lipschitz continuity results for both these functions. For the considered problem Lipschitz continuity is proved.

Chapter 3 concerns problems of parametric optimal control for linear evolution equations and some free boundary problems. In this class of problems control is executed through coefficients of the involved elliptic operators.

Since these optimal control problems may not have solutions a concept of generalized solution is introduced. This cencept is based on the notion of the so called G-convergence of operators.

The results concerning G-convergence of the second order elliptic operators are presented with emphasis on isotropic operators. They are used to define generalized parametric optimal control problems for parabolic equations and variational inequalities. These generalized problems have solutions and for them necessary conditions of optimality are presented.

One of the most important areas, where parametric optimization problems occur in practice is optimal design of mechanical structures.

Chapters 4 and 5 are devoted to numerical methods for solving some optimal design problems.

Chapter 4 deals with optimal design of a plate with respect the fundamental frequency of its free vibrations. The volume of the plate is fixed, while its thickness is subject to optimization. The optimization problem consists in maximizing the smallest eigenvalue of the fourth order elliptic eigenvalue problem describing free vibrations of the plate.

To approximate this problem the finite element method is employed. The convergence of approximation is proved.

The discretized problem is nonsmooth in the case where the smallest eigenvalues are multiple, therefore to solve it a method of nondifferentiable optimization is used. Numerical examples are presented.

In Chapter 5 an optimal shape design problem for two-dimensional elastic body, subject to external forces is investigated. Like in the previous papers by the author the approach used is based on direct minimization of the performance index with respect to some shape parameters treated as decision variables. However, in contrast to the previous papers, perforated domains are considered. A method of homogenization is applied, allowing to approximate the orginal problem with a reasonable accuracy.

A two level design of the shape is proposed and it is illustrated by numerical examples.

Chapter 6 concerns numerical methods for solving multiphase problems of Stefan type in several space variables.

The method exploits the fixed domain formulation of the problems in the form of variational inequalities of the parabolic or mixed elliptic-parabolic type. For this formulation stable approximation schemes are constructed using finite elements in space and finite differences in time variables. The schemes provide a simple time-stepping algorithm. Presented results of numerical experiments indicate the efficiency of the proposed algorithm both for parabolic and for degenerate elliptic-parabolic Stefan problems.

Chapter 7 presents a modification and simplification of an efficient numerical algorithm of solving optimal control problems for systems described by nonlinear ordinary differential equations where the cost functional depends both on the terminal state and the whole trajectory. The original algorithm was developed by the author and Y. Shindo.

Chapter 8 is devoted to effective construction of a quasi-optimal feedback solution for linear differential games, without the necessity of solving the partial differential equation associated with the optimal strategy. The analysis is performed from the point of view of the first player. Both deterministic and stochastic cases are considered.

In the deterministic case construction of a quasi-optimal feedback requires the knowledge of the strategy used by the second player. In this case a numerical example is provided.

In the stochastic case it is allowed that the second player uses nonanticipating processes as the admissible strategies, and it is shown that the analitical form of the quasi-optimal feedback for the first player is independent of the strategy used by the second player.

In Chapter 9 the feedback Nash equilibrium strategies are considered for continuous-time, deterministic two-person differential game with a nonlinear state equation and quadratic cost functionals. The nonlinearity of the state equation appears as a regular perturbation.

The optimal feedback strategy is obtained in the form of a series. The elements of the series can be calculated by solving a matrix Riccati equation and a sequence of quasi-linear partial differential equations.

Several theorems concerning the asymptotic properties of the approximations of the Nash equilibrium strategies are included.

Chapter 10 deals with calculating surrogate constraints mainly for integer programming problems. It is well known that the surrogate dual problems can offer effective bounds on the primal optimal values. Knowledge of these bounds is of a great importance to any branch-and-bound algorithm. However, solving of the dual problems is rather difficult since it requires maximizing of a quasi-concave, often discontinuous, function.

A certain method for calculating surrogate constraints is analysed theoretically and numerically.

The proposed algorithm is based on the concept of the quasi-subgradient generalizing the notion of the subgradient for quasi-concave functions.

The convergence of the algorithm is proved and some numerical results are presented.

CONTENTS

Chapter 1

DIFFERENTIAL STABILITY OF PROJECTION IN HILBERT SPACE ONTO CONVEX SET. APPLICATIONS TO SENSITIVITY ANALYSIS OF OPTIMAL CONTROL PROBLEMS

Jan Sokołowski

1. Introduction

The paper is concerned with the differential stability of solutions of variational inequalities with respect to the parameter. The first part of the paper is devoted to the differential stability of the projection in Hilbert space onto a closed and convex subset. We exploit the notion of the conical differentiability of the projection mapping. Using the results on conical differentiability of the projection we derive the form of the sensitivity coefficient of an optimal control with respect to the parameter for the constrained optimal control problems for distributed parameter systems.

We start with the following examples.

Example 1.1

Let us consider an elementary example of the projection mapping $P_K(.)$ in R onto the set $K=[0,+\infty)$.
In the case we have

$$\forall x \in R \ : \ P_K(x) = x^+ = \begin{cases} x \ , & x \geq 0 \\ 0 \ , & x < 0 \end{cases} \qquad (1.1)$$

It is easy to see that the mapping $x \to x^+$ is differentiable everywhere except at $x=0$. At the point $y=0$ we have for $h=\pm 1$ and for $\varepsilon > 0$:

$$(y + \varepsilon h)^+ = y^+ + \varepsilon h^+ \qquad (1.2)$$

therefore for $\varepsilon > 0$

$$[(y + \varepsilon h)^+ - y^+]/\varepsilon = h^+ = \lim_{\varepsilon \downarrow 0} [(y + \varepsilon h)^+ - y^+]/\varepsilon \qquad (1.3)$$

hence at $y=0$ we have

$$P_K(y + \varepsilon h) = P_K(y) + \varepsilon Q(h) + o(\varepsilon) \qquad (1.4)$$

where the mapping $Q(.) : R \to R$ is defined by $Q(h)=h^+$, $\forall h \in R$.

In the notation of the paper the mapping $Q(.)$ is called the conical differential of the projection $P_K(.)$ at $y=0$.

Let us recall how the projection mapping $P_K(.)$ is related to the variational inequalities. Since for a given $x \in R$ we have

$$(x^+ - x)^2 \leq (v - x)^2, \quad \forall v \in K \tag{1.5}$$

then by a standard argument it follows that the element $x^+=P_K(x)$ is given by unique solution of the following variational inequality:

$$x^+ \in K : (x^+ - x)(v - x^+) \geq 0, \quad \forall v \in K \tag{1.6}$$

■

In this paper we will use the results on differential stability of the projection in a Hilbert space for the local sensitivity analysis of optimal controls to constrained, convex optimal control problems depending on the parameter. Let us show how the differential stability of optimal controls is related to the differential stability of solutions of variational inequalities. We present a simple example of the optimal control problem for ordinary differential equation.

Example 1.2

We denote by $L^2(0,T)$ the space of square integrable functions on $(0,T)$, $T > 0$. $L^2(0,T)$ is Hilbert space with the scalar product

$$(y,z)_{L^2(0,T)} = \int_0^T y(t)\, z(t)dt, \quad \forall y,z \in L^2(0,T) \tag{1.7}$$

We denote by $H^1(0,T)$ Sobolev space:

$$H^1(0,T) = \{\phi \in L^2(0,T) \mid \frac{d\phi}{dt} \in L^2(0,T)\} \tag{1.8}$$

Space $H^1(0,T)$ is Hilbert space with the scalar product

$$(y,z)_{H^1(0,T)} = \int_0^T \{y(t)z(t) + \frac{dy}{dt}(t)\frac{dz}{dt}(t)\}dt \tag{1.9}$$

In order to define an optimal control problem we introduce the state equation, the cost functional and the set of admissible states of the form:

<u>state equation</u>:

$$\frac{dy}{dt}(t) = y(t) + u(t), \quad t \in (0,T)$$
$$y(0) = 0 \tag{1.10}$$

$u(.) \in L^2(0,T)$ denotes control

$y(.) \in H^1(0,T)$ denotes state

<u>cost functional</u>:

$$J(u) = \frac{1}{2}\int_0^T (y(t) - y_d(t))^2 dt + \frac{\alpha}{2}\int_0^T (u(t))^2 dt \tag{1.11}$$

$\alpha > 0$, where $y_d(.) \in L^2(0,T)$ is given element

<u>set of admissible states</u>:

$$Y_{ad} = \{y(.) \in H^1(0,T) \mid y(0)=0, \ a \leq y(T) \leq b\} \tag{1.12}$$

where $a,b \in R$ are given constants.

We denote by $u_o \in L^2(0,T)$ an optimal control which minimizes the cost functional (1.11) subject to state equation (1.10) and state constraints (1.12); we denote by $y_o(.) \in H^1(0,T)$ the optimal state. Let us consider the differential stability of the mapping

$$L^2(0,T) \ni y_d \longrightarrow u_o \in L^2(0,T) \tag{1.13}$$

Let $h(.) \in L^2(0,T)$ be a given element, denote by $u_\varepsilon \in L^2(0,T)$, $\varepsilon \in [0,\delta)$, $\delta > 0$, an optimal control which minimizes the cost functional

$$J_\varepsilon(u) = \frac{1}{2}\int_0^T (y(t) - y_d(t) - \varepsilon h(t))^2 dt + \frac{\alpha}{2}\int_0^T (u(t))^2 dt$$

subject to state equation (1.10) and state constraints (1.12). Denote by $y_\varepsilon \in H^1(0,T)$ the optimal state given by a unique solution of the state equation:

$$\frac{dt_\varepsilon}{dt}(t) = y_\varepsilon(t) + u_\varepsilon(t), \quad t \in (0,T) \tag{1.14}$$

$$y_\varepsilon(0) = 0 \tag{1.15}$$

It can be verified that the optimal state is given by a unique solution of the following variational inequality:

find an element $y_\varepsilon \in K$ such that

$$a(y_\varepsilon, \phi - y_\varepsilon) \geq \int_0^T y_d(t)(\phi(t) - y_\varepsilon(t))dt \qquad \forall \phi \in K \tag{1.16}$$

where $K \overset{\text{def}}{=} Y_{ad}$ and the bilinear form $a(.,.) : H^1(0,T) \times H^1(0,T) \to R$ is defined as follows

$$a(y,z) \overset{\text{def}}{=} \int_0^T \{\alpha\dot{y}(t)\dot{z}(t) - \alpha y(t)\dot{z}(t) - \alpha\dot{y}(t)z(t) + (1+\alpha)y(t)z(t)\}dt\,, \qquad \forall y,z \in H^1(0,T) \tag{1.17}$$

here we denote $\dot{y} = dy/dt$.

We can apply the results on differential stability of the metric projection in Hilbert space presented in the paper to the variational inequality (1.16). It follows that for $\varepsilon > 0$, ε small enough

$$y_\varepsilon = y_0 + \varepsilon z + o(\varepsilon) \quad \text{in} \quad H^1(0,T) \tag{1.18}$$

where $\|o(\varepsilon)\|_{H^1(0,T)}/\varepsilon \to 0$ with $\varepsilon \downarrow 0$. The element $z \in H^1(0,T)$ is given by a unique solution of the following variational inequality:

find an element $z \in S$ such that

$$a(z, \phi - z) \geq \int_0^T h(t)(\phi(t) - z(t))dt\,, \qquad \forall \phi \in S \tag{1.19}$$

where the cone S is given by

$$S = \{\phi \in H^1(0,T) \mid \phi(0) = 0, \tag{1.20}$$

$$\phi(T) \geq 0 \quad \text{if} \quad y_0(T) = a,$$

$$\phi(T) \leq 0 \quad \text{if} \quad y_0(T) = b,$$

$$a(y_0, \phi) = \int_0^T y_d(t)\phi(t)dt\}$$

From (1.18) and (1.14) it follows that for $\varepsilon > 0$, ε small enough:

$$u_\varepsilon = u_0 + \varepsilon q + o(\varepsilon) \quad \text{in} \quad L^2(0,T) \tag{1.21}$$

where $\|o(\varepsilon)\|_{L^2(0,T)}/\varepsilon \to 0$ with $\varepsilon \downarrow 0$.

It can be verified that the element $q \in L^2(0,T)$ is given by a unique solution of the following optimal control problem:

find an element $q \in L^2(0,T)$ which minimizes the cost functional

$$I(u) = \frac{1}{2}\int_0^T (z(t)-h(t))^2 dt + \frac{\alpha}{2}\int_0^T (u(t))^2 dt \tag{1.22}$$

subject to state equation (1.10) and state constraints:

$$z(T) \le 0 \quad \text{if} \quad y_o(T) = b \tag{1.23}$$

$$z(T) \ge 0 \quad \text{if} \quad y_o(T) = a \tag{1.24}$$

$$a(y_o, z) = \int_0^T y_d(t) z(t) dt \tag{1.25}$$

The element q in (1.21) is called the sensitivit coefficient for the optimal control u_o. The Example 1.2 shows that the sensitivity coefficient for an optimal control can be obtained in the form of an optimal solution of the auxiliary optimal control problem.
The differential stability of solutions of variational inequalities with respect to the perturbations of the right-hand side has been studied by Mignot [19] and Haraux [7]. In [19] the notion of a polyhedric convex subset of Hilbert space is introduced and the form of the so-called conical differential of the projection onto such a subset is derived.

Several results on differential stability of metric projection in Hilbert space onto convex set are given by Holmes [8] and by Fitzpatrick and Phelps [5], we refer the reader also to [37] for the related results.

The differential stability of solutions to constrained mathematical programming problems is investigated e.g. in [4, 9, 15]. The results presented in [15] has been used in [16, 17] in order to derive the form of the right-derivatives of solutions to convex, constrained optimal control problems for systems described by ordinary differential equations. Sensitivity analysis of the constrained optimal control problems for partial differential equations is considered in [18] using the similar method as in [16, 17].

In this paper the method proposed by the author [24, 25, 26, 29] based on the conical differentiability of the projection is used in order to derive the form of the right-derivative of an optimal control for optimal control problems for distributed parameter systems with respect to the parameter. In this chapter the right - derivative of an

optimal control is called the sensitivity coefficient of an optimal control with respect to the parameter.
The main result which is used in our method of the sensitivity analysis [24, 26] is the following: the sensitivity coefficient of an optimal solution with respect to the parameter can be derived in the form of an optimal solution of an auxiliary constrained optimization problem.
For further results on differential stability of solutions to variational inequalities as well as on the sensitivity analysis of the optimal control problems we refer the reader to [22, 23, 27, 28]. In [30-36] the applications to the shape sensitivity analysis of free boundary problems are given. The related results on the shape sensitivity analysis of optimal control problems are presented in [25, 26, 29].
The outline of this chapter is following. In Section 2 the projection mapping in Hilbert space onto convex, closed subset is considered. The notion of the conical differentiability of the mapping is introduced.
In Section 3 an abstract result on conical differentiability of the projection mapping is presented. Section 4 is concerned with the differential stability of solutions to an abstract, constrained optimization problem. An example of constrained optimal control problem is provided.
Finally in Section 5 the results on differential stability of optimal controls for two examples are presented.
In the paper the standard notation is used [11]. The related results concerning variational inequalities and optimal control problems can be found in [3, 6, 10, 12, 13, 14, 21].
We use the following notation [11].
Let $\Omega \subset R^n$ be a given domain with the smooth boundary $\Gamma = \partial\Omega$. We denote by $L^2(\Omega)$ the space of square integrable functions on Ω. $L^2(\Omega)$ is Hilbert space with scalar product of the form:

$$(y,z)_{L^2(\Omega)} = \int_\Omega y(x)z(x)dx \, , \quad \forall y,z \in L^2(\Omega) \tag{1.26}$$

We denote by $H^1(\Omega)$, $H^2(\Omega)$ Sobolev spaces:

$$H^1(\Omega) = \{\phi \in L^2(\Omega) \mid \frac{\partial \phi}{\partial x_i} \in L^2(\Omega), \quad i=1,\dots,n\} \tag{1.27}$$

$$H^2(\Omega) = \{\phi \in L^2(\Omega) \mid \frac{\partial \phi}{\partial x_i}, \frac{\partial^2 \phi}{\partial x_i \partial x_j} \in L^2(\Omega), \quad i,j=1,\dots,n\} \tag{1.28}$$

Spaces $H^1(\Omega)$, $H^2(\Omega)$ are Hilbert spaces [11] with the scalar products:

$$(y,z)_{H^1(\Omega)} = \int_\Omega \{y(x)z(x) + \nabla y(x).\nabla z(x)\}dx$$

here we denote $\nabla y(x) = \text{col}\left(\frac{\partial y}{\partial x_1}, \dots, \frac{\partial y}{\partial x_n}\right)$

$$(y,z)_{H^2(\Omega)} = \int_\Omega \{y(x)z(x)+\nabla y(x).\nabla z(x)+\Delta y(x).\Delta z(x)\}dx$$

where $\Delta y = \text{div}(\nabla y) = \sum_{i=1}^{n} \frac{\partial^2 y}{\partial x_i^2}$.

Sobolev space $H_0^1(\Omega)$ is defined as follows [11]:

$$H_0^1(\Omega) = \{\phi \in H^1(\Omega) \mid \phi(x) = 0 \quad \text{on} \quad \partial\Omega\} \tag{1.29}$$

It is Hilbert space with the scalar product:

$$(y,z)_{H_0^1(\Omega)} = \int_\Omega \nabla y(x).\nabla z(x)dx \tag{1.30}$$

2. Projection mapping in Hilbert space

Let H be a separable Hilbert space, $K \subset H$ a convex and closed subset. Let there be given a bilinear form

$$a(.,.) : H \times H \rightarrow R \tag{2.1}$$

which is coercive and continuous i.e.,

$$a(v,v) \geq \alpha \|v\|_H^2, \quad \alpha > 0, \quad \forall v \in H \tag{2.2}$$

$$|a(v,z)| \leq M \|v\|_H \|z\|_H, \quad \forall v,z \in H \tag{2.3}$$

Let H' denotes the dual space of H and let $f \in H'$ be a given element. We denote by $y=P(f)$ a unique solution of the variational inequality:

$$\begin{aligned} &y = P(f) \in K \\ &a(y,v-y) \geq \langle f,v-y \rangle, \quad \forall v \in K \end{aligned} \tag{2.4}$$

where $\langle .,. \rangle$ is the a duality pairing between H' and H.

Remark 2.1:

If the bilinear form $a(.,.)$ is symmetric i.e., $a(v,z)=a(z,v)$, $\forall v,z \in H$ then

$$y = P(f) = \arg\min\{\frac{1}{2} a(v,v) - \langle f,v \rangle \mid v \in K\} \tag{2.5}$$

■

It can be verified that the mapping

$$H' \ni f \longrightarrow P(f) \in H \tag{2.6}$$

is Lipschitz continuous:

$$\|P(f_1) - P(f_2)\|_H \leq \frac{M}{\alpha} \|f_1 - f_2\|_{H'}, \quad \forall f_1, f_2 \in H' \tag{2.7}$$

therefore by a generalization of the Rademacher theorem [19] it follows that there exists a dense subset $\Xi \subset H'$ such that for $f \in \Xi$ we have

$$\forall h \in H' : P(f + \varepsilon h) = P(f) + \varepsilon P'(h) + r(\varepsilon) \quad \text{in} \quad H \tag{2.8}$$

where $r(\varepsilon)/\varepsilon \to 0$ strongly in H with $\varepsilon \to 0$.
The mapping $P'(.) = P'(f;.) : H' \to H$ is linear and continuous. In the sequal we will use the concept of the so-called conical differentiability of the projection operator.

Definition 2.1

The mapping (2.6) is conically differentiable at $f \in H'$ if there exists a continuous mapping

$$Q(.) : H' \longrightarrow H \tag{2.9}$$

such that for $\varepsilon > 0$, ε small enough

$$\forall h \in H' : P(f + \varepsilon h) = P(f) + \varepsilon Q(h) + o(\varepsilon) \quad \text{in} \quad H \tag{2.10}$$

where $\|o(\varepsilon)\|_H/\varepsilon \to 0$ with $\varepsilon \downarrow 0$ uniformly on compact subsets of H'.
In order to derive the form of the mapping (2.9) we need the following notation.

For a given element $y \in K$ we denote by $C_K(y)$ the tangent cone

$$C_K(y) = \{\phi \in H \mid \exists \varepsilon > 0 \text{ such that } y + \varepsilon\phi \in K\} \tag{2.11}$$

In general the cone (2.11) is not closed, we denote by $\overline{C_K(y)}$ its closure in H.