Advanced Mathematics Volume 1

Lin Fanghua and Yang Xiaoping

Geometric Measure Theory
—An Introduction

(几何测度引论)

Science Press
Beijing/New York

International Press
Boston

Lin Fanghua and Yang Xiaoping

Geometric Measure Theory —An Introduction

(几 何 测 度 引 论)

SCIENCE PRESS
Beijing/New York

INTERNATIONAL PRESS
Boston

Sold and distributed in the People's Republic of China by Science Press Beijing

Sold and distributed all over the world with the exception of the People's Republic of China by International Press

Published by

Science Press
16 Donghuangchenggen North Street
Beijing 100717, China

International Press
P. O. Box 2872
Cambridge, MA 02238-2872
U. S. A

ISBN 7-03-010271-1/O-1599(Science Press)
ISBN 1-57146-125-6(International Press)

Series Preface

International Press and Science Press are delighted to bring our readers our joint series of books by outstanding Chinese mathematicians. These books will allow our readers to become acquainted with significant research by the most renowned Chinese scholars, living within China and abroad.

The series will have world wide circulation in both the Chinese and English languages. It is our hope that the series will both encourage many young scholars in China to enter the field of mathematics and also will help further awareness in other parts of the world of important accomplishments by Chinese mathematicians.

We hope that you will appreciate each of these books as much as we do.

Y. Wang and S. T. Yau

Series Editors

Introduction

Since the publication of the seminal work of H. Federer[40] which gives a rather complete and comprehensive discussion on the subject, the geometric measure theory has developed in the last three decades into an even more cohesive body of basic knowledge with an ample structure of its own, established strong ties with many other subject areas of mathematics and made numerous new striking applications. The present book is intended for the researchers in other fields of mathematics as well as graduate students for a quick overview on the subject of the geometric measure theory with emphases on various basic ideas, techniques and their applications in problems arising in the calculus of variations, geometrical analysis and nonlinear partial differential equations. With this intention, the presentation and selection of materials in the book are somewhat different from many other books on the subject, excluding various closed discussions of some special sub-topics dealing with other existing literature. Most similar to this publication is the book written by L. Simon[101] about twenty years ago, aiming at catering to the need of geometrical analysts, PDE specialists and others to master the basic ideas and powerful techniques in the geometric measure theory. Unlike [101], the present text contains many more concrete examples besides the regularity theory of minimal surfaces, illustrating how these ideas and techniques were applied. Indeed, practically each chapter contains such discussions with Chapter 2 in particular. Another important distinction of the present text from [101], although the selection of materials is quite related, is that we have tried to give more detailed expositions of some topics that were either briefly discussed or omitted in [101]. One of the topics we have emphasized is the fundamental notion of *Rectifiability* of sets and measures. Besicovitch[15] laid the foundations of geometric measure theory, particularly, the theory of rectifiable and purely unrectifiable sets by describing to an amazing extent the structure of the subsets of the plane having finite one-dimensional Hausdorff measure. Federer extended Besicovitch's work to m-dimensional subsets of R^n, with m being an integer, and Marstrand analyzed general fractals in the plane whose Hausdorff dimensions need not be an integer and, later studied 2-dimensional sets in R^3. Mattila generalized Marstrand's work for general m-dimensional sets in R^n. Preiss solved one of the most long-standing fundamental open problems which was referred to as Besicovitch-Federer's conjecture, effectively introducing and using tangent measures. It is clear that the study of the rectifiable sets and measures is essential to the theory. Readers should compare various ideas in establishing the rectifiability theorems. In particular, it would be interesting to compare discussions in Chapter 3 for the general theory of rectifiable sets and that of Allard's rectifiability theorem for varifolds in Chapter 6 and Ambrosio-Kirchheim's updated proof of Federer-Flemming's rectifiability theorem for currents in Chapter 7. For many detailed discussions on the rectifiability and the related topics we refer to the recent beautiful treatment by P. Mattila[84]. He also gives much more detailed discussion on rectifiability of measures and Preiss' theorem, rectifiability and analytic capacity, rectifiability and orthogonal projection, rectifiability and singular integrals and many other more traditional topics in the geometric measure theory.

Other topics which have been given much more detailed expositions than those in [101] are functions of bounded variation, sets of finite perimeter and area and co-area formulae. The book[37] by Evans and Gariepy is a wonderful source for the discussions on these topics. Discussions related to BV-functions, sets of finite perimeter and least area oriented boundaries, etc, can be found in the book[51] by E. Giusti which presents a relatively complete account of De Giorgi's theory and his solutions to the classical Plateau's problem.

One of the regrets with respect to the contents of the present text is that it does not contain detailed discussion on fractals and fractal measures. Fractals and fractal measures arise in many ways, for example, in number theory via Diophantine approximation, in probability via Brownian motions and other stochastic processes, in dynamical systems as strange attractors, Julia-sets or some general limit sets of Kleinian groups in complex analysis, in the study of geometry of discrete groups and in many applied analyses, etc. Some discussions on them and further references can be found, for example, in Barnsley[14], Edgar[36], Falconer[38,39], Mandelbrot[79] and Peitgen and Richter[90]. Mandelbrot[79] also uses fractals to model many physical phenomena. Computer simulation of fractal images is widely considered in Peitgen and Saupe[91], and Barnsley[14]. There are also many recent interesting developments in analysis on fractals.

As mentioned above, one of the emphases of the present text is to include many applications. It is obvious that our discussions on applications are by no means exclusive. In fact, we have only chosen a few simple examples to illustrate several important ideas and techniques in the geometric measure theory. We wish to draw readers' attention to a few recent important works whose spirits, ideas, techniques are very much related to part of our discussions here. Among them we would like to point out

(a) L. Caffarelli's work on the study of regularity of free boundaries as well as singular sets of free boundaries[25,26,27];

(b) J. Cheeger and T. Colding's work on Riemannian manifolds with nonnegative Ricci curvature[30];

(c) L. Simon's work on singular sets of energy minimizing harmonic maps or area-minimizing currents[103–109];

(d) some recent works on stationary harmonic maps[75], Yang-Mills fields[117,118], Seiberg-Witten's equations[112–114] and Ginzburg-Landau equations in high dimensions[76,77].

The present book does not deal with the relationships between the classical harmonic analysis and the geometric measure theory. Several interesting works by G. David and S. Semmes, the monograph[31] in particular, may offer readers some aspects of such theory. We should point out that the later chapters of [84] also relate the issues of this type. The recent work by Kenig and Toro[63] on harmonic measures on Reifenberg flat domains gives another fascinating application of the theory.

We now briefly describe the topics of each chapter.

In Chapter 1, we introduce one of the most important measures, the Hausdorff measure, in the geometric measure theory along with several related notions such as the Hausdorff distance, and the Hausdorff dimensions. Some other measures are discussed at the end of the chapter. The main aim of this chapter is to illustrate the covering technique. By using both Vitali and Besicovitch covering lemmas, we establish density properties of sets and the relation between Lebesgue measure and the Hausdorff measures. Due to concerning densities, an effort has also been made to introduce tangent measures and to establish the Marstrand's theorem.

The entire Chapter 2 is devoted to the applications related to the Hausdorff measures. We first show the Federer-Zimmer and Calderon-Zygmund's theorem concerning the Lebesgue points and differentiable points of Sobolev functions. Some further discussions can be found in the books[37,122]. We then establish the partial regularity theorem for energy minimizing harmonic maps into spheres. Using the Hausdorff metric, we show Blaschke's selection principle and Almgren's δ- regularity theorem. We then prove the Federer's dimension reduction principle and then explain many applications in the studies of nodal and critical sets, homogenizations in partial differential equations, maps from Alexandroff geometry, etc. At the end of the chapter we discuss the Reifenberg's topological disc theorems and some recent works by Kenig and Toro on Reifenberg-flat domains applying Preiss' idea.

In Chapter 3, we study Lipschitz functions and rectifiable sets. We show the extension theorem, differentiability theorem and C^1 approximation theorem for Lipschitz functions. A basic rectifiability theorem is established under the assumption that there exists almost everywhere a (unique) approximate tangent space. Marstrand and Mattila's rectifiability theorems under the weak tangent plane properties (nonuniqueness of tangent spaces) are also established. An effort is also made to explain the deep theorem of D. Preiss concerning Besicovitch-Federer's conjecture. We prove that 'density one' implies the rectifiability and develop the key structure theorem which characterizes rectifiable sets by their projection properties.

Chapter 4 is devoted to the detailed proofs of the area and coarea formulae. Some applications are discussed including the degree theory for VMO mappings developed recently by Brezis and Nirenberg[23].

In Chapter 5, we study the set of the finite perimeter and functions of bounded variations. After establishing various basic facts for BV-functions as those of Sobolev functions, we also prove the coarea formula for BV functions. De Giorgi's theorem concerning the set of finite perimeter is also proved. Some further discussions are made at the end of the chapter about the class of special BV functions.

Chapter 6 is the basic varifold theory. We first explain the idea of the Young measures that will naturally lead to the notion of generalized surfaces—varifolds. Then we introduce the notion of varifolds and their first variation. The basic monotonicity lemma and isoperimetric inequality are established for the general varifolds which have controlled the first variations. The basic rectifiability and regularity theorem of Allard are then explained. Further discussions on these can be found in [101]. Here we simply present Allard's main ideas.

In Chapter 7, we discuss some fundamental results concerning integral rectifiable currents due to Federer-Fleming, including deformation theorem, compactness theorem and rectifiability theorem. The updated proofs here are mainly taken from a recent article of Ambrosio and Kirchheim[13]. Federer-Fleming's proof (with some improvements) was nicely explained in the book by L. Simon[101](see also [40]).

Finally in Chapter 8, we discuss existence and De Giorgi's theorem concerning the regularity of area-minimizing oriented boundaries. The key argument is how to establish the excess decay and height bound. The proof we adopt is taken from [55]. De Giorgi's original proof is discussed in detail in the book by E. Giusti[51]. The proof given in [55] can be viewed as simplification from the earlier works by Almgren[6–9], Schoen-Simon[96] and Bomberi[19].

The present book is essentially self-contained. A sufficient prerequisite for reading this book is to have some knowledge of real analysis (real-variable and measure theory), Sobolev spaces and differential geometry.

Geometric measure theory is a hard subject. Needless to say, the present text must contain many defects and authors' ignorance. We simply hope that it will provide a brief account of the theory, some basic ideas and techniques for research beginners and many others interested in these subjects.

The authors

July 2001

Acknowledgements

This book grows out of the lectures given by the first author at the summer school of the Mathematical Institute of Zhejiang University, Hangzhou, China, during 1997–1999. The lecture notes were taken, refined and revised by the second author. Chapter 7 contains several important results concerning integral rectifiable currents. This part of lectures was given by Professor Robert Hardt. He has spent a great deal of time to explain various relatively recent proofs of these results. We wish to express our cordial thanks here to him for his inspiring lectures.

Parts of the lectures were based on a set of notes from an introductory course on geometric measure theory given by the first author at the Courant Institute of Mathematical Sciences. We wish to thank the former Ph.D students Han Qing, Clara-Hume Padilla, and Yoshihero Tonegawa supervised by the first author for helping with this early set of notes. The summer school was greatly supported by several people. Among them, we wish to thank, in particular, Professors Chen Shuping, Hong Jiaxing, Guo Boling, K. C. Chang, and Miao Changxing for their supports and encouragement. We wish also to thank the Secretary at the Mathematical Institute of Zhejiang University, Ms. Shen for numerous assistance. The financial supports of the summer school were mainly from some summer school grants given by the National Natural Science Foundation of China (NSFC) as well as the first author's B-Class award for young outstanding researchers. The authors would like to express their gratitude to them for their supports. The success of the summer school also largely depends on a big group of young inspiriting audience. We thank them for their interests in the course. Last but not least, the first author wishes to thank his wife, Wu Xi, for her understanding and supports over all these years. The financial supports of the second author come from a project of the NSFC. The second author wishes to thank his wife, Cai Ping, for her constant supports.

Lin Fanghua
Yang Xiaoping
July 2001

Contents

Chapter 1
Hausdorff Measure

Modern measure theory can go back to C. Jordan (1893) who founded the so-called Jordan measure theory. E. Borel (1895) invented a new method for studying measures of point sets which guaranteed the denumerable additivity of measures. R. Baire soon after published the first paper of discontinuous real function theory. Largely inspired by these stimulating work, Lebesgue (1902), who was the first to set forth systematically the idea on measure and integration, introduced a kind of measure $\mathcal{L}^n$ over R^n (Lebesgue measure) which is the generalization of length, area, volume, etc. Caratheodory (1914) proposed to define measures by set covering. Hausdorff (1919) adopted this idea to define fractional dimensional measure (Hausdorff measure). In the middle of this century, Besicovitch and his students laid down the groundwork and a series of deep properties of Hausdorff measures.

In this chapter, we will concentrate mainly on the elementary theory of Hausdorff measure. In the first section we briefly review some basic theory of outer measure and define the Hausdorff measure. The second section focuses on the proof of isodiametric inequality involving the Steiner symmetrization. The fact that symmetrization in coordinate directions only will be sufficient to show such inequality is observed by L. C. Evans. The third section includes the covering theory and densities for Hausdorff measures. Many techniques in Geometric Measure Theory involve covering arguments. We will illustrate this in proofs of density theorems and the theorems in Section 2.1 later. At the end of the third section we also explain tangent measures and prove one beautiful theorem due to Marstrand. The final section presents some more general measures and extensions related to the Hausdorff measure, including integral geometric measure, net measure, the Banach Paradox, etc. Besides the Hausdorff measure, one of the most important measures in Geometric Measure Theory is probably the integral geometric measure.

1.1 Preliminaries, Definitions and Properties

In this section, we first quickly recall some elementary measure theory, then give the definition of Hausdorff measure and its fundamental properties.

Let X be a topological space. All subsets of X form a family of subsets of X denoted by $\mathcal{S}$. A collection $\mathcal{C}$ of subsets of X is said to be a *σ-algebra* if

(i) $\emptyset$, $X \in \mathcal{C}$.

(ii) $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$ and $\bigcap_{i=1}^{\infty} A_i \in \mathcal{C}$ whenever $A_i \in \mathcal{C}$ for $i = 1, 2, 3, ...,$

(iii) $X \sim A \in \mathcal{C}$ whenever $A \in \mathcal{C}$.

Borel subsets of X are the elements of the smallest family of subsets of X which contains the open and closed subsets of X and is closed under the formation of countable unions and intersections. The collection $\mathcal{B}$ of all Borel subsets of X is a σ-algebra. Now we recall the definition of (outer) measures.

1.1.1 Definition. *Let X be a topological space. If μ is a function: $\mathcal{S} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and*

$$\mu(A) \leq \sum_{j=1}^{\infty} \mu(A_j)$$

whenever $A \subset \bigcup_{j=1}^{\infty} A_j$, $A_j \subset \mathcal{S}$, we say that μ is a measure over X or a measure in brief.

A subset $A \subset X$ is called to be μ-measurable if for each subset $B \subset X$,

$$\mu(B) = \mu(B \sim A) + \mu(B \cap A).$$

It is usually referred to as the Caratheodory approach or the Caratheodory condition. It is easy to show that the subfamily of $\mathcal{S}$ consisting of all μ-measurable subsets is a σ-algebra.

A measure μ on X is called a *Borel measure* if each Borel set is μ-measurable. A Borel measure μ is called *Borel regular* if for each subset $A \subset X$ there exists a Borel set $B \supset A$ such that $\mu(B) = \mu(A)$.

1.1.2 Theorem (Caratheodory Criterion). *For a measure μ on a metric space X, all open sets are μ-measurable if and only if*

$$\mu(A \cup B) \geq \mu(A) + \mu(B)$$

whenever $A \subset X$, $B \subset X$ and $dist(A, B) > 0$.

Proof. The necessity is obvious. To show the sufficient part it is enough to prove that for any subset $T \subset X$, and each open subset $O \subset X$, there holds

$$\mu(T) \geq \mu(T \sim O) + \mu(T \cap O)$$

whenever $\mu(T) < \infty$. In fact, set $T \cap O = C$ and

$$C_k = C \cap \left\{ x \in X, \quad \text{dist}(x, X \sim O) \geq \frac{1}{k} \right\}.$$

Then $\text{dist}(C_k, T \sim O) > 0$ and

$$\mu(T) \geq \mu(T \cap C_k) + \mu(T \sim O).$$

Now we claim that

$$\lim_{k \to \infty} \mu(T \cap C_k) = \mu(T \cap C) = \mu(T \cap O),$$

which is equivalent to $\lim_{k\to\infty} \mu(C_k) = \mu(C)$.

Let $C_0 = \emptyset$, $R_n = C_{n+1} \sim C_n$, and employ the condition of this part to obtain that

$$\mu(C_{2k+1}) \geq \sum_{n=1}^{k} \mu(R_{2n}).$$

and

$$\mu(C_{2k}) \geq \sum_{n=1}^{k} \mu(R_{2n-1}).$$

Observe that

$$C = \bigcup_{k=1}^{\infty} C_k = C_{2k} + \bigcup_{n=k}^{\infty} R_{2n} + \bigcup_{n=k+1}^{\infty} R_{2n-1}$$

and

$$\sum_{n=1}^{\infty} \mu(R_{2n}) \leq \mu(T) < \infty.$$

$$\sum_{n=1}^{\infty} \mu(R_{2n+1}) \leq \mu(T) < \infty.$$

So

$$\mu(C) \leq \mu(C_{2k}) + \sum_{n=k}^{\infty} \mu(R_{2n}) + \sum_{n=k+1}^{\infty} \mu(R_{2n-1}).$$

This implies that $\lim_{k\to\infty} \mu(C_k) = \mu(C)$ as required. □

Suppose that μ is a Borel regular measure on X and $X = \bigcup_{i=1}^{\infty} U_i$ where U_i is open and $\mu(U_i) < \infty$ for each $i = 1, 2, 3, \ldots$. Then we can check that

$$\mu(A) = \inf_{O \text{ open}, \, O \supset A} \mu(O)$$

for each subset $A \subset X$, and

$$\mu(A) = \sup_{C \text{ closed}, \, C \subset A} \mu(C)$$

for each μ-measurable subset $A \subset X$.

If the topological space X is locally compact and separable, the terminology that a measure μ is a *Radon measure* means that μ is Borel regular and finite on each compact subset of X.

Let U be an open, bounded smooth subset of R^n. $\mathcal{M}(U)$ denotes the space of signed Radon measures on U with finite mass. $C_0(U)$ is the space of continuous, real-valued functions on U with compact supports.

1.1.3 Definition. *A sequence $\{\mu_k\}_{k=1}^{\infty} \subset \mathcal{M}(U)$ is said to converge weakly to $\mu \in \mathcal{M}(U)$, denoted by $\mu_k \rightharpoonup \mu$ weakly in $\mathcal{M}(U)$, if*

$$\int_U f d\mu_k \to \int_U f d\mu \text{ as } k \to \infty \quad \text{for each } f \in C_0(U).$$

1.1.4 Theorem. *Assume that $\mu_k \rightharpoonup \mu$ weakly in $\mathcal{M}(U)$. Then*

$$\limsup_{k \to \infty} \mu_k(C) \le \mu(C)$$

for each compact set $C \subset U$ and

$$\mu(O) \le \liminf_{k \to \infty} \mu_k(O)$$

for each open set $O \subset U$.

Proof. For any $\epsilon > 0$, by the definition of measure, we see that there exist a $\delta > 0$ and a δ-neighbourhood C_δ of C such that

$$\mu(C_\delta) \le \mu(C) + \epsilon.$$

Choose $f(x) = \min\{1, \frac{1}{\delta}\text{dist}(x, U - C_\delta)\}$ and infer that

$$\mu(C) + \epsilon > \int f d\mu = \lim_{k \to \infty} \int f d\mu_k \ge \limsup_{k \to \infty} \mu_k(C).$$

On the other hand if $m < \mu(O)$, $A \subset O$ is a compact set with $\mu(A) > m$ and $\delta > 0$ satisfies that $A_\delta \subset O$, we define f similarly as above to obtain

$$m < \int f d\mu = \lim_{k \to \infty} \int f d\mu_k \le \liminf_{k \to \infty} \mu_k(O).$$

Obviously these complete the proof. □

The metrics on $\mathcal{M}(U)$ are very useful. We briefly recall some facts in the aspects.

1.1.5 Definition. *By saying that a sequence of measures $\mu_n \in \mathcal{M}(U)$ converges to a measure μ in the Levi metric ρ we mean that for any $\epsilon > 0$, there exists an $N > 0$ such that*

$$\rho(\mu_n, \mu) < \epsilon \text{ whenever } n \geq N,$$

where $\rho(\mu, \nu) < \epsilon$ if and only if for any $\delta > 0$, the δ-neighborhood A_δ of A, there hold

$$\mu(A) \leq \nu(A_\delta) + \epsilon, \;\; \nu(A) \leq \mu(A_\delta) + \epsilon.$$

1.1.6 Lemma. *The Levi metric convergence in Definition 1.1.5 is equivalent to the measure's weak convergence in Definition 1.1.3.*

The proof of this lemma is standard, we leave it as an exercise.

One of fundamental theorems in measure theory is the Riesz Representation Theorem for linear functionals. To recall it we assume that X is locally compact and separable, and H is a Hilbert space with inner product $\langle , \rangle$. $L(X, H)$ denotes the space of all continuous functions from X to H with compact supports. Then we have the following statement (see [101] for the proof).

1.1.7 Theorem (Riesz Representation Theorem). *Let $\mathcal{F}$ be any linear functional on $L(X, H)$ satisfying*

$$\sup\{\mathcal{F}(f) : f \in L(X, H), |\, f \,| \leq 1, \operatorname{spt} f \subset C\} < \infty$$

for each compact $C \subset X$. Then there is a Radon measure μ on X and a μ-measurable function $\nu : X \to H$ such that $|\, \nu(x) \,| = 1$ for μ-a.e. $x \in X$ and

$$\mathcal{F}(f) = \int_X \langle f, \nu \rangle d\mu \text{ for all } f \in L(X, H).$$

We now begin to discuss some lower dimensional measures on R^n.

1.1.8 Definition. *Let $A \subseteq R^n$, $0 \leq s < \infty$.*

(i)*For $0 \leq \delta \leq \infty$, define*

$$H^s_\delta(A) := \inf\left\{ \sum_{j=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam} C_j}{2}\right)^s \Big| A \subset \bigcup_{j=1}^{\infty} C_j, \operatorname{diam} C_j \leq \delta \right\},$$

where

$$\alpha(s) = \frac{\pi^{s/2}}{\Gamma(\frac{s}{2} + 1)}, \quad 0 \leq s < \infty,$$

and $\Gamma(s) =: \int_0^\infty e^{-x}x^{s-1}dx$, $0 < s < \infty$, *that is,* $\Gamma(s)$ *is the (real)* Γ-*function.*

(ii) *Define*

$$H^s(A) := \lim_{\delta\to 0} H^s_\delta(A) = \sup_{\delta>0} H^s_\delta(A).$$

We call H^s *s-dimensional Hausdorff measure on* R^n.

Remark 1. For $\delta > \delta'$, $H^s_\delta(A) \le H^s_{\delta'}(A)$. Hence $H^s_\delta(\cdot)$ is a monotonously decreasing function of $\delta \in [0, \infty]$.

Remark 2. H^1_δ is not a Borel measure on R^2.

Remark 3. We will occasionally make reference to s-dimensional spherical measure, defined for A, s, δ as above in the following way

$$S^s_\delta(A) := \inf\left\{\sum_{j=1}^\infty \alpha(s) r_j^s | 0 < 2r_j < \delta, A \subset \bigcup_{j=1}^\infty B(x_j, r_j)\right\},$$
$$S^s(A) := \lim_{\delta\to 0} S^s_\delta(A).$$

In general, $H^s \le S^s \le 2^s H^s$; however, there exist such sets A so that they are H^s-measurable in R^n with $0 < H^s(A) < \infty$ and $H^s(A) < S^s(A)$. For instance, Besicovitch constructed a set $A \subset R^2$ with $S^1(A) = 2/\sqrt{3}$, $H^1(A) = 1$(see [40]).

1.1.9 Theorem. H^s *is a Borel regular measure* $(0 \le s < \infty)$.

Proof. (i) We show that H^s is a measure. That is, for $\{A_n\}_{n=1}^\infty \subset R^n$,

$$H^s\left(\bigcup_{k=1}^\infty A_k\right) \le \sum_{k=1}^\infty H^s(A_k).$$

By definition, the inequality can be easily established for H^s_δ $(\delta > 0)$. Then we take limit $\delta \to 0$ to arrive at the conclusion.

(ii) We prove that H^s is a Borel measure.

One has to show that if A, $B \subset R^n$ with $\text{dist}(A, B) > 0$, at least either A or B is a Borel set, then

$$H^s(A \cup B) = H^s(A) + H^s(B).$$

It is easy to prove by definition that, if $\text{dist}(A, B) > 3\delta$, then

$$H^s_\delta(A \cup B) = H^s_\delta(A) + H^s_\delta(B).$$

Then letting $\delta \to 0^+$, one obtains the desired result.

(iii) We show that H^s is regular.