HANDBOOK *of* NUMERICAL ANALYSIS

P. G. CIARLET *and* J. L. LIONS • Editors

Volume

I

Finite Difference Methods (Part 1)

Solution of Equations in $\mathbf{R}^n$ (Part 1)

NORTH-HOLLAND

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General Preface

During the past decades, giant needs for ever more sophisticated mathematical models and increasingly complex and extensive computer simulations have arisen. In this fashion, two indissociable activities, *mathematical modeling* and *computer simulation*, have gained a major status in all aspects of science, technology, and industry.

In order that these two sciences be established on the safest possible grounds, mathematical rigor is indispensable. For this reason, two companion sciences, *Numerical Analysis* and *Scientific Software*, have emerged as essential steps for validating the mathematical models and the computer simulations that are based on them.

Numerical Analysis is here understood as the part of *Mathematics* that describes and analyzes all the numerical schemes that are used on computers; its objective consists in obtaining a clear, precise, and faithful, representation of all the "information" contained in a mathematical model; as such, it is the natural extension of more classical tools, such as analytic solutions, special transforms, functional analysis, as well as stability and asymptotic analysis.

The various volumes comprising the *Handbook of Numerical Analysis* will thoroughly cover all the major aspects of Numerical Analysis, by presenting accessible and in-depth surveys, which include the most recent trends.

More precisely, the Handbook will cover the *basic methods of Numerical Analysis*, gathered under the following general headings:

- Solution of Equations in $\mathbb{R}^n$,
- Finite Difference Methods,
- Finite Element Methods,
- Techniques of Scientific Computing,
- Optimization Theory and Systems Science.

It will also cover the *numerical solution of actual problems of contemporary interest in Applied Mathematics*, gathered under the following general headings:

- Numerical Methods for Fluids,
- Numerical Methods for Solids,
- Specific Applications.

"Specific Applications" include: Meteorology, Seismology, Petroleum Mechanics, Celestial Mechanics, etc.

Each heading is covered by several *articles*, each of which being devoted to a specialized, but to some extent "independent", topic. Each article contains a thorough description and a mathematical analysis of the various methods in actual use, whose practical performances may be illustrated by significant numerical examples.

Since the Handbook is basically expository in nature, only the most basic results are usually proved in detail, while less important, or technical, results may be only stated or commented upon (in which case specific references for their proofs are systematically provided). In the same spirit, only a "selective" bibliography is appended whenever the roughest counts indicate that the reference list of an article should comprise several thousands items if it were to be exhaustive.

Volumes are numbered by capital Roman numerals (as Vol. I, Vol. II, etc.), according to their *chronological appearance*.

Since all the articles pertaining to a given *heading* may not be simultaneously available at a given time, a given heading usually appears in more than one volume; for instance, if articles devoted to the heading "Solution of Equations in $\mathbb{R}^n$" appear in Volumes I and III, these volumes will include "Solution of Equations in $\mathbb{R}^n$ (Part 1)" and "Solution of Equations in $\mathbb{R}^n$ (Part 2)" in their respective titles. Naturally, all the headings dealt with within a given volume appear in its title; for instance, the complete title of Volume I is "Finite Difference Methods (Part 1)—Solution of Equations in $\mathbb{R}^n$ (Part 1)".

Each article is subdivided into *sections*, which are numbered consecutively throughout the article by *Arabic numerals*, as Section 1, Section 2, ..., Section 14, etc. Within a given section, *formulas*, *theorems*, *remarks*, and *figures*, have their own independent numberings; for instance, within Section 14, formulas are numbered consecutively as (14.1), (14.2), etc., theorems are numbered consecutively as Theorem 14.1, Theorem 14.2, etc. For the sake of clarity, the article is also subdivided into *chapters*, numbered consecutively throughout the article by *capital Roman numerals*; for instance, Chapter I comprises Sections 1 to 9, Chapter II comprises Sections 10 to 16, etc.

P.G. CIARLET
J.L. LIONS
May 1989

Finite Difference Methods (Part 1)

Introduction

G.I. Marchuk

The finite difference method is a universal and efficient numerical method for solving differential equations. Its intensive development, which began at the end of 1940s and the beginning of 1950s, was stimulated by the need to cope with a number of complex problems of science and technology. Powerful computers provided an impetus of paramount importance for the development and application of the finite difference method which in itself is sufficiently simple in utilization and can be conveniently realized using computers of different architecture. A large number of complicated multidimensional problems in electrodynamics, elasticity theory, fluid mechanics, gas dynamics, theory of particle and radiation transfer, atmosphere and ocean dynamics, and plasma physics were solved employing the finite difference techniques.

Numerous spectacular results have been obtained in the theory of finite difference methods during the last four decades.

In ordinary differential equations, the stability of the main classical finite difference methods was investigated and the relevant accuracy estimates were constructed, a large number of new versions of these methods were constructed, and efficient algorithms were suggested for their realization in a wide field of applications-oriented problems. The needs of electronics, kinetics, and catalysis stimulated the development of a broad class of methods for solving stiff systems of equations. Problems in control theory, biology, and medicine were important for the progress in finite difference methods of solving delay ordinary differential equations.

In partial differential equations, the achievements of the finite difference method are even more impressive. Finite difference counterparts of the main differential operators of mathematical physics were constructed, including those with conservation properties, that is, those obeying the discrete counterparts of the laws of conservation. An elegant theory of approximation, stability, and convergence of the finite difference method was constructed.

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The efforts of the specialists in differential equations and numerical mathematics yielded a convenient and efficient apparatus of the finite difference method, including spectral analysis, discrete maximum principle, and energy method. Considerable progress was achieved in the methods on a sequence of grids, including extrapolation methods. A flood of new results in the theory of the finite element method greatly stimulated new achievements in the theory of the finite difference method.

An important stage in the progress of finite difference methods was the development of the alternating direction implicit method, the fractional steps method, and the splitting method. The realization of these methods consists in solving a large number of one-dimensional problems. A considerable number of versions of this class of methods have been suggested, having high approximation accuracy and absolute stability. Numerous multidimensional problems in physics, mechanics, and geophysical hydrodynamics have been solved using these methods.

The theory of the finite difference method is far from having been completed, especially in the field of nonlinear partial differential equations. Life never ceases to offer new complex problems, and the method of finite differences remains a powerful approach to solving them.

It would be impossible in the *Handbook of Numerical Analysis* to cover even all the basic achievements of the theory. Nevertheless, I do not doubt that its publication will help making the finite difference method interesting to new people and will attract new researchers to solving its problems.

Finite Difference Methods for Linear Parabolic Equations

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Contents

Preface

This article is devoted to the numerical solution of linear partial differential equations of parabolic type by means of finite difference methods. The emphasis in our presentation will be on concepts and basic principles of the analysis, and we shall therefore often restrict our considerations to model problems in as much as the choice of the parabolic equation, the regularity of its coefficients, the underlying geometry and the boundary conditions are concerned. In the beginning of the article proofs are provided for the principal results, but as the situation under investigation requires more technical machinery, the analysis will become more sketchy. The reader will then be referred to the literature quoted for fuller accounts of both results and theoretical foundations.

The article is divided into three chapters. The first of these is of an introductory nature and presents some of the basic problems and concepts of the theory for the model heat equation in one space dimension. This chapter is subdivided into two sections, devoted to the pure initial value problem and the mixed initial boundary value problem, respectively. This division is then the basis for the plan of the rest of the article, where Chapters II and III treat these two classes of problems in greater depth and generality. Whereas most of the theory in Chapter II depends on Fourier analysis, that of Chapter III relies heavily on energy and monotonicity type arguments.

The finite difference method for partial differential equations has a relatively short history. After the fundamental theoretical paper by COURANT, FRIEDRICHS and LEWY [1928] on the solution of the problems of mathematical physics by means of finite differences, the subject lay dormant till the period of, and immediately following, the Second World War, when considerable theoretical progress was made, and large scale practical applications became possible with the aid of computers. In this context a major role was played by the work of von Neumann, partly reported in O'BRIAN, HYMAN and KAPLAN [1951]. For parabolic equations a highlight of the early theory was the celebrated paper by JOHN [1952], which had a great influence on the subsequent research. The field then had its golden age during the 1950s and 1960s, and major contributions were given by Douglas, Kreiss, Lees, Samarskii, Widlund and others.

At the end of this period the theory for the pure initial value problem had become reasonably well developed and complete, and this was essentially also true for mixed initial boundary value problems in one space dimension. For multidimensional problems in general domains the situation was less satisfactory, partly because the finite difference method employs the values of the solution at the points of a uniform mesh, which does not necessarily fit the domain. This led the development into a different direction based on variational formulations of the boundary value problems, and using piecewise polynomial approximating functions on more flexible partitions of the domain. This approach, the finite element method, was better suited for complex geometries and many numerical analysts (including the author of this article) abandoned the classical finite difference method to work with finite elements. The papers on parabolic equations using finite differences after 1970 are few, particularly in the West.

It should be said, however, that finite elements and finite differences have many points in common, and that it may be more appropriate to think of the new development as a continuation of the established theory rather than a break away from it. In the Russian literature, for instance, finite element methods are often referred to as variational difference schemes, and variational thinking was, in fact, used already in the paper by Courant, Friedrichs and Lewy quoted above. The finite element theory owes much of its present level of development and sophistication to the foundation provided by the finite difference theory. However, in the present Handbook, the two subjects are separated, and we shall only very briefly touch upon their interrelation below.

In our presentation here we shall not discuss techniques for solving the algebraic linear systems of equations that result from the discretization of the initial boundary value problems, but refer to other articles of this Handbook concerning such matters. Neither shall we treat the related area of alternating direction implicit methods, or fractional step methods, which are designed to reduce the amount of computation needed in multidimensional, particularly rectangular, domains, and to which a special article of this volume is devoted.

Several textbooks exist which treat finite difference methods for parabolic problems, and we refer, in particular, to Richtmyer and Morton [1967] and Samarskii and Gulin [1973] for thorough accounts of the field, but also (in chronological order of publication) to Collatz [1955], Forsythe and Wasow [1960], Rjabenki and Filippow [1960], Fox [1962], Godunov and Ryabenkii [1964], Saul'ev [1964], Smith [1965], Babuška, Práger and Vitásek [1966], Mitchell [1969], and Samarskii [1971]. In addition we would like to mention the survey papers by Douglas [1961a] and Thomée [1969]. We have included in our list of references a large number of original papers, not all of which are quoted in our text. For treatises on the theory of parabolic differential equations, covering existence, uniqueness and regularity results such as needed here, we refer to Friedman [1964] and Ladyženskaja, Solonnikov and Ural'ceva [1968].

I would like to take this opportunity to thank Chalmers University of Technology for granting me a reduction of my teaching load while writing this article, to Ann-Britt Karlsson and Yumi Karlsson for typing the manuscript, and to Mohammad Asadzadeh for proofreading the entire work.

CHAPTER I

Introduction

In this first introductory chapter our purpose is to use the simplest possible model problems to present some basic concepts which are important for the understanding of the formulation and analysis of finite difference methods for parabolic partial differential equations. The chapter is subdivided into two sections corresponding to the two basic problems discussed in the rest of this article, namely the pure initial value problem and the mixed initial boundary value problem.

In the first section we thus consider the pure initial value problem for the heat equation in one space dimension. We begin with the simplest example of an explicit one-step, or two-level, finite difference approximation, discuss its stability with respect to the maximum norm and relate its formal accuracy to its rate of convergence to the exact solution. We also present an example of the construction of a more accurate explicit scheme. We then introduce the application of Fourier techniques in the analysis of both stability, now with respect to the L_2-norm, accuracy, and convergence. We finally touch upon the possibility of using more than two time levels in our approximations.

Section 2 is devoted to the mixed initial boundary value problem for the same basic parabolic equation, with Dirichlet type boundary conditions at the endpoints of a finite interval in the space variable. Here we discuss the possibility and advantage of using implicit methods, requiring the solution of a linear system of equations at each time level. Stability and error analysis is carried out for the simplest such methods, the backward Euler method and the more accurate Crank–Nicolson method. Again both maximum-norm estimates based on positivity properties of the difference scheme and l_2-norm estimates derived by Fourier analysis are treated. A brief mention is made of the possibility of extending some initial boundary value problems to periodic pure initial value problems.

The material in this chapter is standard and we refer to the basic textbooks quoted in our preface for further details and references.

1. The pure initial value problem

In this first section of our introduction to the solution of parabolic problems by means of finite difference methods we shall discuss several such methods for the pure initial value problem for the simple homogeneous heat equation in one space dimension.

We thus wish to find the solution of the pure initial value problem

$$\begin{aligned} &\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \quad \text{for } x\in\mathbb{R},\quad t\geqslant 0,\\ &u(x,0)=v(x) \quad \text{for } x\in\mathbb{R}, \end{aligned} \tag{1.1}$$

where $\mathbb{R}$ denotes the real axis and v is a given smooth bounded function. It is well known that this problem admits a unique solution, many properties of which may be deduced, for instance, from the representation

$$u(x,t)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \mathrm{e}^{-(x-y)^2/(4t)}v(y)\,\mathrm{d}y\equiv (E(t)v)(x).$$

Here we think of the right-hand side as defining the solution operator $E(t)$ of (1.1). In particular, we may note that this solution operator is bounded, or, more precisely,

$$\sup_{x\in\mathbb{R}}|E(t)v(x)|=\sup_{x\in\mathbb{R}}|u(x,t)|\leqslant\sup_{x\in\mathbb{R}}|v(x)| \quad \text{for } t>0. \tag{1.2}$$

For the numerical solution of the problem (1.1) by finite differences we introduce a grid of mesh points $(x,t)=(jh,nk)$ where h and k are mesh parameters which are small and thought of as tending to zero, and where j and n are integers, $n\geqslant 0$. We then look for an approximate solution of (1.1) at these mesh points, which will be denoted by U_j^n, by solving a problem in which the derivatives in (1.1) have been replaced by finite difference quotients. Define thus for functions defined on the grid the forward and backward difference quotients

$$\begin{aligned} &\partial_x U_j^n = h^{-1}(U_{j+1}^n - U_j^n),\\ &\bar\partial_x U_j^n = h^{-1}(U_j^n - U_{j-1}^n), \end{aligned}$$

and similarly, for instance,

$$\partial_t U_j^n = k^{-1}(U_j^{n+1}-U_j^n).$$

The simplest finite difference equation corresponding to (1.1) is then

$$\begin{aligned} &\partial_t U_j^n=\partial_x\bar\partial_x U_j^n \quad \text{for } -\infty<j<\infty,\quad n\geqslant 0,\\ &U_j^0=v_j\equiv v(jh) \quad \text{for } -\infty<j<\infty. \end{aligned}$$

This difference equation may also be written as

$$\frac{U_j^{n+1}-U_j^n}{k}=\frac{U_{j+1}^n-2U_j^n+U_{j-1}^n}{h^2},$$

or, if we set $\lambda=k/h^2$,

$$U_j^{n+1}=\lambda U_{j-1}^n+(1-2\lambda)U_j^n+\lambda U_{j+1}^n\equiv(E_{kh}U^n)_j, \tag{1.3}$$

where the identity defines a linear operator E_{kh}, the local discrete solution operator. This scheme is called explicit since it expresses the solution at $t=(n+1)k$ explicitly in terms of the values at $t=nk$. Iterating the operator we find that the solution of the