Systems and Transforms with Applications in Optics

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SYSTEMS AND TRANSFORMS WITH APPLICATIONS IN OPTICS

PREFACE

In recent years, a trend has been developing toward greater interaction between electrical engineering and optics. This is so not only because optical devices are used extensively in signal processing, storage, pattern recognition, and other areas, but also because the underlying theory is closely related to the theory of systems, transforms, and stochastic processes. In fact, whereas in system analysis the Fourier integral is an auxiliary concept, in diffraction theory it represents a physical quantity; whereas only a limited class of electrical signals need be treated as stochastic processes, optical waves are inherently random. The following list illustrates the striking parallels between these two disciplines.

Fresnel diffraction: output of a filter with quadratic phase
Fraunhofer field: Fourier transform
Lens: linear FM generator
Focal plane field: Fourier transform
Contrast improvement: filtering

Apodization: pulse shaping
Coherence: autocorrelation
Michelson interferometer: correlometer consisting of a delay line and an adder
Fabry-Perot interferometer: narrow-band filter

Motivated by such observations, I decided to develop a course in the general area of systems and optics. In the early planning stages of the course, two approaches seemed attractive: (1) descriptive coverage of a large number of applications, or (2) extension of signal theory to two dimensions and space coordinates followed by analytical discussion of selected applications. As is apparent from this book, I chose the second approach. This choice reflects my conviction that in education the primary objective is not the exhaustive coverage of terminal topics but rather the systematic analysis of all steps from first principles to representative illustrations.

The material in this book is essentially self-contained. Part 1 deals with the general theory of systems and transforms in one and two dimensions. The early notions of the one-dimensional case are covered only lightly; greater emphasis is placed on more sophisticated concepts, some of which are new. The two-dimensional case includes singularity functions, systems, Hankel transforms viewed as Fourier transforms, sampling expansions, asymptotic expansions, and stochastic processes. Part 2 is devoted to optical applications related to the material in Part 1. Diffraction theory is based entirely on Kirchhoff's formula, thin lenses are interpreted as transparencies with quadratically varying phase, and the study of coherence is an extension of the second-order theory of random signals. To avoid any prerequisites from electromagnetic theory, I considered only the scalar theory of light. The required background from the theory of stochastic processes is given in Chapter 8.

In exploring the relationship between systems and optics, I had the following objectives in mind: To make available to our students an important area and to present it in a language with which they are familiar; to introduce into optics a point of view that simplifies and unifies a number of apparently unrelated topics; and to point out certain analogies that facilitate the transfer of knowledge from one field to the other. As an interesting example of such analogies, I mention the pulse compression technique used in radar. This rather recent idea is equivalent to the old principle of concentration of light by a lens.

I should like to emphasize that my aim here is not to formulate a general mathematical theory but rather to develop certain analytical techniques and to show their relevance in a large number of applications. For this reason, I derive various results only formally, often ignoring mathematical subtleties. This is apparent in the sections on singularity functions and asymptotic expansions, and in the proofs of various theorems in Fourier analysis.

Most topics covered in the book have been treated elsewhere; however, the approach is distinctly different. Related references are listed in the Bibliography and in several footnotes. Any omissions are due to my ignorance. The book "Principles of Optics" by Born and Wolf deserves special mention.

In the planning and execution of this project I enjoyed the complete understanding of the then department head (at the Polytechnic Institute of Brooklyn), Rudy Drenick. It is my pleasure to express to him my appreciation. I also thank my colleagues Leonard Bergstein and Lawrence Levey for their helpful suggestions and critical comments.

Athanasios Papoulis

CONTENTS

1
SYSTEMS AND TRANSFORMS IN DIFFRACTION THEORY

In Part 1 of this book we develop the basic concepts of systems and transforms in one variable, and we extend the investigation to two dimensions. In the selection of topics we were influenced by optical applications. The treatment is self-contained; however, the emphasis is placed not on detailed discussion of early concepts, but on an integrated development of a variety of applications, some of them new. Some familiarity with the elementary theory of one-dimensional systems and transforms is, therefore, desirable.

In the first chapter, we outline certain areas of optics in which the notions of systems, Fourier transforms, and stochastic signals are used extensively. Since this material is included mainly for motivation, most theorems are presented briefly, often without proof. All results of this chapter will be reestablished in Part 2.

1. FOURIER TRANSFORMS IN OPTICS

Consider a monochromatic plane wave

$$v^i(x,y,z,t) = f(x,y)e^{-j\omega(t-z/c)} \tag{1-1}$$

propagating in the z direction with velocity† **c**. If an opaque screen with an aperture S is placed on the $z = 0$ plane (Fig. 1-1), then from geometric optics one would deduce that the space $z > 0$ is divided into a "visible" cylindrical region, where the field is given by (1-1), and a "dark" region, where the field is zero. This notion of a beam of light is an approximation that is valid only near the aperture, and the object of diffraction theory is to determine more accurately the diffracted field

$$v(x,y,z,t) = g(x,y,z)e^{-j\omega t} \tag{1-2}$$

at large distances.‡ It can be shown that the field $g(x_o,y_o,z_o)$ at a point $P:(x_o,y_o,z_o)$ of the right space is given by

$$g(x_o,y_o,z_o) \simeq A \iint_S f(x,y)e^{jkr}\,dx\,dy \qquad k = \frac{\omega}{\mathbf{c}} \tag{1-3}$$

where $f(x,y)$ is the function in (1-1) specifying the incident wave v^i, and

$$r = \sqrt{(x - x_o)^2 + (y - y_o)^2 + z_o^2} \tag{1-4}$$

is the distance from P to the point of integration $(x,y,0)$. The coefficient A is a slowly varying function of the coordinates of P. At the end of the

† The boldface letter **c** will be used to denote the velocity of light.
‡ A. Marechal and M. Françon, "Diffraction," Paris, 1960.

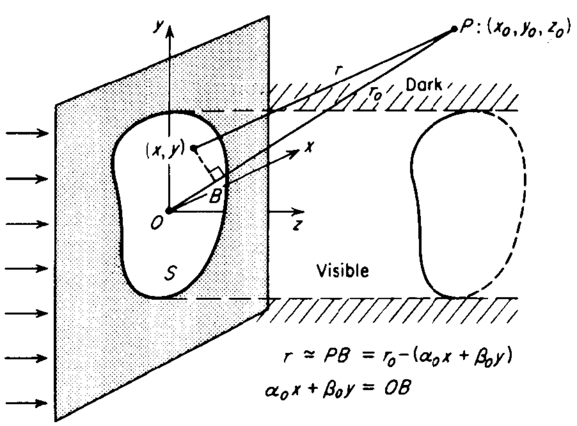

Fig. 1-1

section we trace briefly the steps from first principles to the approximate relationship (1-3).

FRAUNHOFER APPROXIMATION

With r_o the distance from the origin to the point P and

$$\alpha_o = \frac{x_o}{r_o} \qquad \beta_o = \frac{y_o}{r_o}$$

the directional cosines of the line OP, we see from Fig. 1-1 that if r_o is large compared with the dimensions of S [see (1-17)], then

$$r \simeq r_o - (\alpha_o x + \beta_o y) \tag{1-5}$$

and (1-3) yields

$$g(x_o,y_o,z_o) = Ae^{jkr_o} \iint_S f(x,y)e^{-jk(\alpha_o x+\beta_o y)}\,dx\,dy \tag{1-6}$$

This result can be expressed in terms of the two-dimensional Fourier transform

$$F(u,v) = \iint_{-\infty}^{\infty} f(x,y)e^{-j(ux+vy)}\,dx\,dy \tag{1-7}$$

of the function $f(x,y)$. Since $f(x,y) = 0$ for (x,y) not in S, (1-6) takes the form

$$g(x_o,y_o,z_o) = Ae^{jkr_o}F(k\alpha_o,k\beta_o) \tag{1-8}$$

Thus, on the surface of a sphere centered at the origin, the *amplitude*† of the diffraction field is proportional to the Fourier transform of the aperture function $f(x,y)$.

On the plane $z = z_o$ this is not the case because r_o is no longer constant. However, in a small-angle region $\alpha_o = x_o/r_o \simeq x_o/z_o$, $\beta_o = y_o/r_o \simeq y_o/z_o$, and

$$|g(x_o,y_o,z_o)| = \left| AF\left(\frac{kx_o}{z_o}, \frac{ky_o}{z_o}\right) \right| \tag{1-9}$$

Example 1-1 From (1-9) it follows that if S is a square with sides $2a$ and $2b$, and $f(x,y) = 1$, then [see Chap. 3, (4-30)]

$$|g(x_o,y_o,z_o)| = \left| \frac{4A \sin(kax_o/z_o) \sin(kby_o/z_o)}{(kx_o/z_o)(ky_o/z_o)} \right| \tag{1-10}$$

† In the following, the term *amplitude* will mean the function $g(x,y,z)$ in (1-2) (the expression *complex amplitude* is also used). The quantity $|g|^2$ will be referred to as the *intensity* of v.

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Hankel transforms

We now assume that S is a circle and $f(x,y)$ has circular symmetry:

$$f(x,y) = f(r) \qquad r = \sqrt{x^2 + y^2}$$

It can be shown that [see Chap. 5, (1-4)] in this case, $F(u,v)$ also has circular symmetry, that is,

$$F(u,v) = F(w) \qquad w = \sqrt{u^2 + v^2}$$

and it is given by

$$F(w) = 2\pi \int_0^a rf(r)J_o(wr)\,dr \tag{1-11}$$

where $J_o(x)$ is the Bessel function of order zero, and a is the radius of S. The function $\bar{f}(w) = F(w)/2\pi$ is the Hankel transform of $f(r)$. Thus [see (1-8)],

$$g(x_o,y_o,z_o) = Ae^{jkr_o}F\left(k\,\frac{\sqrt{x_o{}^2 + y_o{}^2}}{z_o}\right) \tag{1-12}$$

Example 1-2 If the incident wave is uniform, then $f(r) = 1$; hence [see Chap. 5, (1-20)],

$$F(w) = 2\pi \int_0^a rJ_o(rw)\,dr = \frac{2\pi aJ_1(aw)}{w}$$

where $J_1(x)$ is the Bessel function of order one. Inserting into (1-12), we obtain the familiar Airy pattern (Fig. 1-2).

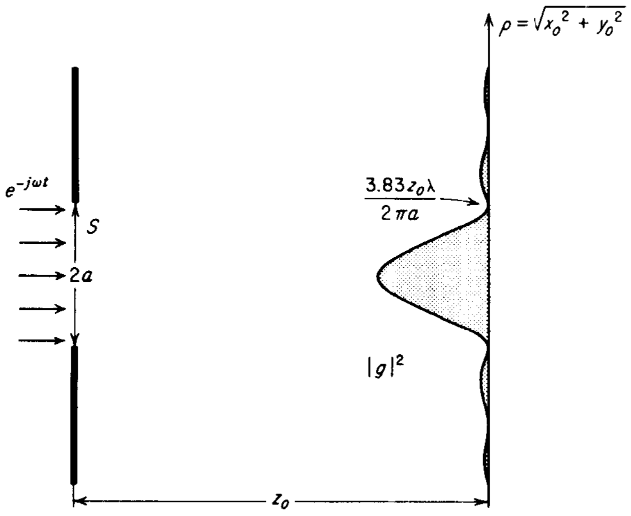

Fig. 1-2

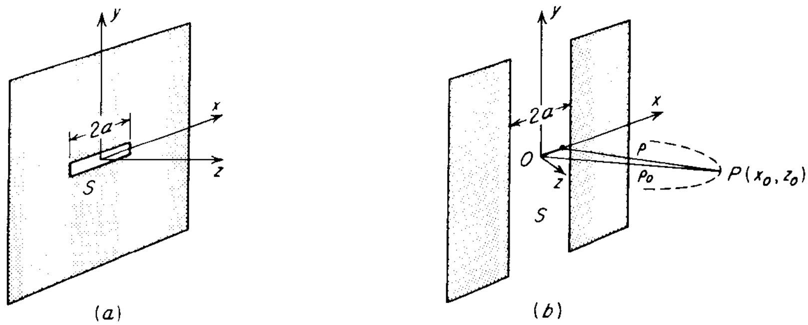

Fig. 1-3

One-dimensional transforms

The Fraunhofer diffraction reduces to the one-dimensional Fourier transform in the following two cases.

Slit apertures If the aperture S is a narrow slit along the x axis of length $2a$ (Fig. 1-3a), then, with

$$f(x) = f(x,0)$$

the incoming wave along the slit and

$$F(u) = \int_{-a}^{a} f(x)e^{-jux}\,dx$$

its Fourier transform, the diffraction field is proportional to $F(k\alpha_o)$ on the surface of a sphere of radius $r_o \gg a$.

Two-dimensional signals We now assume that the incoming wave is independent of y, that is,

$$f(x,y) = f(x)$$

and that the aperture S is an infinite strip along the y axis of width $2a$ (Fig. 1-3b). We can no longer use (1-3) to determine the resulting diffraction field because this formula is based on the assumption that the dimensions of S are small compared with r_o. In our case, the field $g(x,z)$ is independent of y and it can be determined from the two-dimensional analog of (1-3). With

$$\rho = \sqrt{(x - x_o)^2 + z_o^2} \tag{1-13}$$

the distance from the point P to the point $(x,0)$, it can be shown, as in

(1-3), that

$$g(x_o,z_o) = B\int_{-a}^{a} f(x)e^{jk\rho}\,dx \tag{1-14}$$

Introducing the approximation

$$\rho \simeq \rho_o - \alpha_o x \qquad \rho_o = \sqrt{x_o^2 + y_o^2} \qquad \alpha_o = \frac{x_o}{\rho_o}$$

we conclude from (1-14) that

$$g(x_o,z_o) = Be^{jk\rho_o}\int_{-a}^{a} f(x)e^{-jk\alpha_o x}\,dx = Be^{jk\rho_o}F(k\alpha_o) \tag{1-15}$$

Thus, on the surface of a *cylinder*, $g(x_o,z_o)$ is proportional to the Fourier transform of $f(x)$.

Example 1-3 If the incoming wave is uniform, then $f(x) = 1$, and (1-15) yields

$$|g(x_o,z_o)| = \left|\frac{2B\sin(kax_o/z_o)}{kx_o/z_o}\right|$$

Comments

1. If the aperture S is covered with a thin film, then the far field is again given by (1-6), provided that $f(x,y)$ is replaced by the product

$$f(x,y)T(x,y)$$

where $T(x,y)$ is the transmission function characteristic of the film.
2. It can be shown that if a self-luminous two-dimensional object is placed on the $z = 0$ plane and its amplitude (surface source density) equals

$$f(x,y)e^{-j\omega t}$$

then the resulting far field is proportional to $F(k\alpha_o,k\beta_o)$, as in (1-8).
3. As we see from the first two examples, the far field takes significant values only in the cone

$$\frac{\sqrt{x_o^2 + y_o^2}}{z_o} \leq \frac{\lambda}{a} \tag{1-16}$$

where $\lambda = 2\pi\mathbf{c}/\omega$ is the wavelength of the incoming signal, and a is the radius of the smallest circle enclosing S (Fig. 1-4). The above somewhat arbitrary condition gives only the order of magnitude of the angle of the "visible cone." Thus, if

$$a = 10^{-3}\text{ m} \qquad \text{and} \qquad \lambda = 6 \times 10^{-7}\text{ m}$$

then this angle is about 2 minutes. In general, there is no simple relationship between the extent of a function and its Fourier transform. However, from the uncertainty principle (Chap. 6, Sec. 3) it follows that

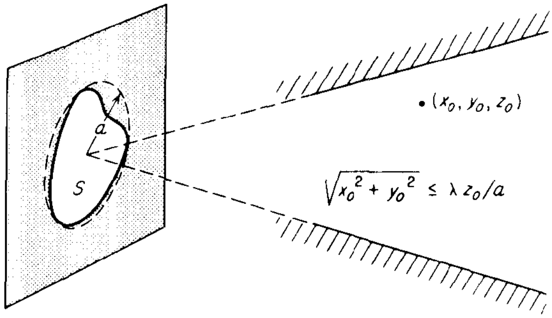

Fig. 1-4

the visible region contains in any case the cone (1-16), and if $f(x,y)$ varies rapidly, it might be considerably broader.

4. It can be shown (Chap. 9, Sec. 3) that the error in (1-6) due to the approximation (1-5) is negligible for

$$z_o > \frac{100a^2}{\lambda} \tag{1-17}$$

Thus, if

$$a = 10^{-3} \text{ m} \qquad \text{and} \qquad \lambda = 5 \times 10^{-7} \text{ m}$$

then the Fraunhofer approximation certainly holds for $z_o > 200$ m.

KIRCHHOFF'S FORMULA AND KIRCHHOFF'S APPROXIMATION

We shall now justify the diffraction integrals (1-3) and (1-14). Suppose that a function $g(x,y,z)$ satisfies the homogeneous wave equation

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} + k^2 g = 0 \tag{1-18}$$

everywhere in the region $z \geq 0$. It can be shown that if $z_o > 0$, then with r as in (1-4),

$$g(x_o,y_o,z_o) = \frac{1}{4\pi} \iint_{-\infty}^{\infty} \left[g \frac{\partial}{\partial z}\left(\frac{e^{jkr}}{r}\right) - \frac{e^{jkr}}{r}\frac{\partial g}{\partial z}\right] dx\, dy \tag{1-19}$$

(provided that g satisfies certain general conditions at infinity). This remarkable formula, expressing $g(x_o,y_o,z_o)$ in terms of g and its normal derivative $\partial g/\partial z$ on the plane $z = 0$, shows that all information needed to determine the propagation of an optical wave is contained on a plane separating the sources from the point of observation. It is a special case of Kirchhoff's formula, which is discussed in Chap. 9, Sec. 1.

To apply the above to the diffraction problem, we observe that since there are no sources in the region $z \geq 0$, the diffracted field (1-2) satisfies the wave equation (1-18); hence, it is given by the integral (1-19). The boundary values g and $\partial g/\partial z$ are, of course, not known; therefore (1-19) cannot be used directly. However, for optical signals, it is reasonable to assume that at the opening S of the screen, the perturbed field v equals the incident field v^i [see (1-1)], that is,

$$g(x,y,0) = f(x,y) \qquad \frac{\partial g}{\partial z}(x,y,0) = jkf(x,y) \qquad (x,y) \in S \tag{1-20}$$

and at the dark side of the screen it equals zero. Inserting into (1-19), we thus conclude that

$$g(x_o,y_o,z_o) = \frac{1}{4\pi} \iint_S f(x,y) e^{jkr} \left[\left(-\frac{1}{r^2} + \frac{jk}{r} \right) \frac{\partial r}{\partial z} - \frac{jk}{r} \right] dx\, dy \tag{1-21}$$

If we assume that z_o is large compared with the dimensions of S, then the terms in the bracket can be approximated by constants

$$r \simeq r_o = \sqrt{x_o^2 + y_o^2 + z_o^2} \qquad \frac{\partial r}{\partial z} = -\frac{z_o}{r} \simeq -\frac{z_o}{r_o}$$

and with

$$A = \frac{1}{4\pi} \left[\left(\frac{1}{r_o^2} - \frac{jk}{r_o} \right) \frac{z_o}{r_o} - \frac{jk}{r_o} \right] \simeq -\frac{jk(1 + z_o/r_o)}{4\pi r_o} \tag{1-22}$$

(1-3) follows. The last approximation resulted because r_o is large compared with the wavelength $\lambda = 2\pi/k$.

For two-dimensional signals, (1-19) reduces to [see Chap. 9, (1-17)]

$$g(x_o,z_o) = \frac{e^{j\pi/4}}{\sqrt{8\pi k}} \int_{-\infty}^{\infty} \left[g \frac{\partial}{\partial z} \left(\frac{e^{jk\rho}}{\sqrt{\rho}} \right) - \frac{e^{jk\rho}}{\sqrt{\rho}} \frac{\partial g}{\partial z} \right] dx \tag{1-23}$$

Introducing Kirchhoff's approximation, we obtain, as in (1-21),

$$g(x_o,z_o) = \frac{e^{j\pi/4}}{\sqrt{8\pi k}} \int_{-a}^{a} f(x) e^{jk\rho} \left[\left(-\frac{1}{2\sqrt{\rho^3}} + \frac{jk}{\sqrt{\rho}} \right) \frac{\partial \rho}{\partial z} - \frac{jk}{\sqrt{\rho}} \right] dx \tag{1-24}$$

If $z_o \gg a$, then the bracket is approximately constant, and with

$$B = \frac{e^{j\pi/4}}{\sqrt{8\pi k}} \left[\left(\frac{1}{2\sqrt{\rho_o^3}} - \frac{jk}{\sqrt{\rho_o}} \right) \frac{z_o}{\rho_o} - \frac{jk}{\sqrt{\rho_o}} \right] \simeq \frac{e^{-j\pi/4}}{2\sqrt{\lambda \rho_o}} \left(1 + \frac{z_o}{\rho_o} \right) \tag{1-25}$$

(1-14) follows.