MATHEMATICS
IN SCIENCE
AND
ENGINEERING

*Volume 134*

# Qualitative Analysis of Large Scale Dynamical Systems

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# *QUALITATIVE ANALYSIS OF LARGE SCALE DYNAMICAL SYSTEMS*


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ACADEMIC PRESS New York San Francisco London 1977

*A Subsidiary of Harcourt Brace Jovanovich, Publishers*

ACADEMIC PRESS, INC.
111 Fifth Avenue, New York, New York 10003

*United Kingdom Edition published by*
ACADEMIC PRESS, INC. (LONDON) LTD.
24/28 Oval Road, London NW1

**Library of Congress Cataloging in Publication Data**

Michel, Anthony N
Qualitative analysis of large scale dynamical systems.

(Mathematics in science and engineering series ; vol. 000)
Bibliography: p.
Includes index.
1. System analysis. I. Miller, Richard K., joint author. II. Title. III. Series.
QA402.M48 515'.7 76-50399
ISBN 0-12-493850-7

PRINTED IN THE UNITED STATES OF AMERICA

# *Preface*

The demands of today's technology have resulted in the planning, design, and realization of sophisticated systems that have become increasingly large in scope and complex in structure. It is therefore not surprising that over the past decade or more, many researchers have directed their attention to various problems that arise in connection with systems of this type, which are called large scale systems. Although it is reasonable to assume that in the near future there will evolve a well-defined body of knowledge on large systems, the directions of such a discipline have not been entirely resolved at this time. However, there are several well-established areas that have reached a reasonable degree of maturity. One is concerned with the qualitative analysis of large scale dynamical systems, the topic of this monograph.

There are numerous examples of large dynamical systems that provide great challenges to engineers of all disciplines, physical scientists, life scientists, economists, social scientists, and of course, applied mathematicians. Obvious examples of large scale dynamical systems include electric power systems, nuclear reactors, aerospace systems, large electric networks, economic systems, process control systems in the chemical and petroleum industries, different types of societal systems, and ecological systems. Most systems of this type have several general properties in common. They may often be viewed as an interconnection of several subsystems. (For this reason, such systems are often also called interconnected systems or composite systems.) In addition, such systems are usually endowed with a complex interconnecting structure and are frequently of high dimension.

In order that this monograph be applicable to many diverse areas and disciplines, we have endeavored to consider several important classes of equations that can be used in the modeling of a great variety of large scale dynamical systems. Specifically, we consider systems that may be represented by ordinary differential equations, ordinary difference equations, stochastic differential equations, functional differential equations, Volterra integrodifferential equations, and certain classes of partial differential equations. In addition, we consider hybrid dynamical systems, which are appropriately modeled by a mixture of different types of equations. Qualitative aspects of large scale dynamical systems that we consider include Lyapunov stability (stability, asymptotic stability, exponential stability, instability, and complete instability), Lagrange stability (boundedness and ultimate boundedness of solutions), estimates of trajectory behavior and trajectory bounds, input–output properties of dynamical systems (input–output stability, i.e., boundedness and continuity of the input–output relations that characterize dynamical systems), and questions concerning the well-posedness of large scale dynamical systems.

The qualitative analysis of large scale systems can be accomplished in a variety of ways. We present a unified approach of analyzing such systems at different hierarchical levels, namely, at the subsystem structure and interconnecting structure levels. This method of analysis offers several advantages. As will be shown, the method of analyzing complex systems in terms of lower order and simpler subsystems and in terms of system interconnecting structure often makes it possible to circumvent difficulties that usually arise in the analysis of high-dimensional systems with intricate structure. We shall also see that this method of analysis is somewhat universal in the sense that it may be applied to all the types of equations enumerated above. It will be seen that this approach is especially well suited for the qualitative analysis of hybrid dynamical systems (i.e., systems described by a mixture of different types of equations). In addition, analysis by this procedure yields trade-off information between qualitative effects of subsystems and interconnection components. This method of analysis also makes it possible to compensate and stabilize large systems at different hierarchical levels, making use of local feedback techniques. Furthermore, this method can be used as a guide in the planning of decentralized systems endowed with built-in reliability (i.e., safety) features. Because of these advantages, this method of analysis should be considered as being more important than the individual results presented. Indeed, all the subsequent results should be viewed as models; in particular applications, one should tailor the present method of analysis to specific problems.

This book consists of seven chapters. In the first chapter we provide an overview of the subject. Chapters II–V are concerned with Lyapunov stability, Lagrange stability, estimates of trajectory behavior and trajectory bounds, and questions of well-posedness of large scale systems. In Chapter II we consider systems described by ordinary differential equations, in Chapter III systems that can be represented by ordinary difference equations and also sampled data systems, and in Chapter IV systems that can be modeled by stochastic differential equations. In Chapter V we address ourselves to infinite-dimensional systems that can be represented by differential equations defined on Banach and Hilbert spaces. Such systems include those that can appropriately be described by functional differential equations, Volterra integro-differential equations, certain classes of partial differential equations, and infinite-dimensional hybrid systems described by a mixture of equations. Chapters VI and VII are devoted to input–output stability properties of large scale dynamical systems. The results in Chapter VI are rather general, while Chapter VII is confined to systems described by integrodifferential equations.

To demonstrate the usefulness of the method of analysis advanced and to point to various advantages and disadvantages, we have included several specific examples from diverse areas, such as problems from control theory, circuit theory, nuclear reactor dynamics, and economics. Because of their importance in applications, we have emphasized frequency domain techniques in several examples.

In order to make this book reasonably self-contained, we have included necessary background material on the following topics: the principal results from the Lyapunov stability theory (for finite-dimensional systems, infinite-dimensional systems, and systems described by stochastic differential equations), the main results for boundedness and ultimate boundedness of solutions, the principal comparison theorems (the comparison principle), results from the theory of M-matrices, selected results from semigroup theory, and pertinent results from systems theory (relating to input–output stability). In addition, we have provided numerous references for this background material.

# *Acknowledgments*

We would like to thank Professor Richard Bellman for encouraging us to undertake this project, and we feel privileged to have this book published in his distinguished series. Likewise, thanks are due to the staff of Academic Press for advice and assistance.

A great part of this monograph is based on research conducted at Iowa State University by the authors and by former students of the first author, Drs. Eric L. Lasley, David W. Porter, and Robert D. Rasmussen. The work of both authors was supported in part by the National Science Foundation, and the first author was also supported by the Engineering Research Institute, Iowa State University. During 1972–1973, the first author's research was accomplished on a sabbatical leave at the Technical University of Graz, Austria, where he had the privilege of being associated with Professor Wolfgang Hahn.

We are particularly appreciative of the efforts of Mrs. Betty A. Carter in typing the manuscript. Above all, we would like to thank our wives, Leone and Pat, for their patience and understanding.

# *Contents*

CHAPTER I

# *Introduction*

In this chapter we first discuss somewhat informally the motivation for the method of analysis advanced in this monograph. We then briefly indicate the type of qualitative analysis with which we concern ourselves. This is followed by an overview of qualitative results for large scale systems which will be of interest to us. Finally, we give an indication of the contents of the subsequent chapters.

## 1.1 Introduction

In recent years many researchers have addressed themselves to various problems concerned with large systems. This is evidenced by an increasing number of publications in scientific journals and conference proceedings. At this time there also have appeared a number of monographs dealing with various aspects of large scale systems. For example, there is the fundamental work by Kron [1] on diakoptics, Tewarson [1] considers the theory of sparse matrices arising naturally in large systems, Lasdon [1] addresses himself to optimization theory of large systems, Mesarović, Macko, and Takahara [1] and Mesarović and Takahara [1] develop a general systems theory for hierarchical multilevel systems, and so forth. Although the state of the art in the area of qualitative analysis of large systems has reached a reasonable degree of maturity, no text summarizing the important results of this topic has appeared. We address ourselves in this book to this problem. As was pointed out in an editorial by Bellman [4], problems associated with large systems offer new and interesting challenges to researchers. We hope that the present monograph will in a small way further stimulate work in this new and exciting field.

It must be stated at the outset that no precise definition of large scale system can be given since this term has a different meaning to different workers. In this book we consider a dynamical system to be large if it possesses a certain degree of complexity in terms of structure and dimensionality. More specifically, we will be interested in dynamical systems which may be viewed as an interconnection of several lower order subsystems. This point of view motivates also the terms "composite system," "interconnected system," "multiloop system," and the like. In certain applications, the term "decentralized system" has also been used.

Roughly speaking, problems concerned with large scale systems may be divided into two broad areas: static problems (e.g., graph theoretic problems, routing problems) and dynamical problems. The latter may in turn be separated into quantitative problems (e.g., numerical solution of equations describing large systems) and into qualitative problems. All topics of this book are concerned with qualitative analysis of large scale dynamical systems.

The traditional approach in systems theory is to represent systems in certain "standard" or canonical forms. For example, the usual approach in classical as well as in modern control theory is to transform the equations describing a given system in such a fashion that the system in question may be represented, for example, in the familiar block diagram form of Fig. 1.1. Once this is accomplished, long-established and well-tested methods are

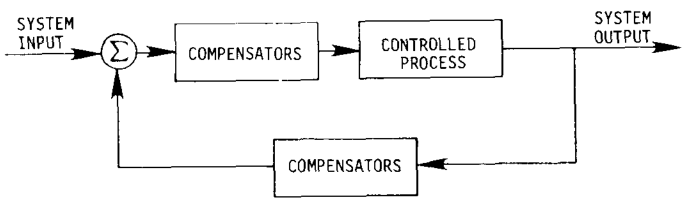


**Figure 1.1** Typical feedback control system.

employed to treat the problem on hand. Frequently, however, certain complications arise with this approach. For example, often difficulties are encountered in analysis and synthesis procedures which can be attributed to the dimensionality and to structural complexity of a system. Another disadvantage of the traditional approach is that after the system in question has been cast into a standard mold (such as Fig. 1.1), the effects of individual components, subloops, etc., are often no longer explicitly apparent because several transformations had to be performed to put the system into a desired canonical form (e.g., Fig. 1.1). In the subsequent chapters we will develop a unified qualitative theory for large systems, in which the objective will always be the same: to analyze (and synthesize) large scale systems in terms of their lower order (and hopefully simpler) subsystems and in terms of their interconnecting structure. In this way, complications which usually arise in the qualitative analysis of high order systems with complex interconnecting structure may often be circumvented. Furthermore, such an approach provides insight into system structure in its original form, yielding information on effects of individual system components, subsystems or subloops, trade-off information between various subsystems and interconnecting structure, and the like. This type of information is usually of great value to the designer and to the analyst. In addition, the viewpoint advanced herein makes it often possible to compensate large systems (i.e., improve their performance) at several hierarchical levels (i.e., at the subsystem level and interconnecting structure level), using local feedback methods. A typical large scale system, in its original form, may conceptually be represented as shown in the block diagram of Fig. 1.2.

The method of analysis advanced herein is of course not without disadvantages. Thus, if a system is decomposed into too many subsystems, one may obtain overly conservative results. However, we will demonstrate by means of several specific examples that this need not be the case, provided that the method is applied properly.

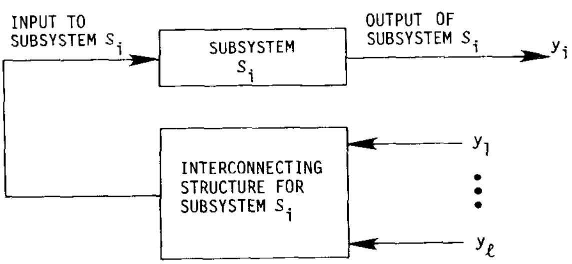


**Figure 1.2** Typical large scale dynamical system where $i = 1, \ldots, l$.

## 1.2 Qualitative Analysis of Dynamical Systems: General Remarks

Qualitative aspects of dynamical systems which we will primarily address ourselves to include the following: stability and instability in the sense of Lyapunov; boundedness of solutions (i.e., Lagrange stability); estimates of trajectory behavior and trajectory bounds; and input–output properties of dynamical systems.

The direct method of Lyapunov has found a wide range of applications in engineering and in the physical sciences. Although most of these applications were originally concerned with the stability analysis of systems described by ordinary differential equations (see, e.g., Hahn [1, 2], LaSalle and Lefschetz [1], Kalman and Bertram [1]), the direct method of Lyapunov has also been extended to systems described by difference equations (see, e.g., Hahn [1, 2], Kalman and Bertram [2]), and more recently to systems represented by partial differential equations (see, e.g., Zubov [1], Wang [1, 5], Sirazetdinov [1], Chaffee [1], Hahn [2]), differential difference equations (see, e.g., Bellman and Cooke [1], Yoshizawa [1], Krasovskii [1], Hahn [2]), functional differential equations (see, e.g., Hale [2] Yoshizawa [1], Krasovskii [1], Hahn [2], Halanay [1]), integrodifferential equations (see, e.g., Driver [1], Levin [1], Miller [1], Suhadloc [1], Bronikovski, Hall and Nohel [1]), systems of countably infinite many ordinary differential equations (see, e.g., Bellman [1], Shaw [1]), stochastic differential equations (see, e.g., Kushner [1], Arnold [1], Kozin [1], Kats and Krasovskii [1], Bertram and Sarachik [1]), stochastic difference equations (see, e.g., Kushner [3]), and the like. The stability theory of general dynamical systems is still of current interest (see, e.g., Bhatia and Szegö [1], Hale [1], Krein [1], Ladas and Lakshmikantham [1], Walter [1], Slemrod [1]).

In the case of many systems (e.g., systems exhibiting nonlinear oscillations) it is not the Lyapunov stability or instability that is of interest. Yoshizawa and others (see, e.g., Yoshizawa [1], Hahn [2], LaSalle and Lefschetz [1]) have extended the direct method of Lyapunov to establish conditions for boundedness (Lagrange stability), ultimate boundedness, unboundedness, and the like, of solutions of dynamical systems.

In practice one is not only interested in the qualitative type of information obtainable from the Lyapunov stability and the Lagrange stability of a dynamical system, but also in specific estimates of trajectory behavior and trajectory bounds. A system could for example be stable or bounded and still be completely useless because it may exhibit undesirable transient characteristics (e.g., its solutions may exceed certain limits or specifications imposed by the designer on the trajectory bounds). Estimates of trajectory behavior and trajectory bounds have been obtained using the Lyapunov-type approach by defining stability with respect to time-varying subsets (of the state space) which are *prespecified* in a given problem (this is not the case in Lyapunov and Lagrange stability). The boundaries of these sets yield estimates of system trajectory behavior and trajectory bounds (see, e.g., Matrosov [1, 2], Michel [1–3], Michel and Heinen [1–4]). This concept includes the notions of practical stability and finite time stability (see, e.g., LaSalle and Lefschetz [1], Weiss and Infante [1], Michel and Porter [4]).

In a radical departure from the classical approach described thus far, it is possible to view dynamical systems in a "black box" sense as relations mapping system inputs into system outputs on an extended function space. In the context of this formulation, the qualitative analysis of such systems is accomplished in terms of system input–output properties (see, e.g., Sandberg [2, 8], Zames [3, 4], Willems [1], Desoer and Vidyasagar [1], Holtzman [1]). Many important results have been obtained along these lines and for many important problems a connection between input–output stability and Lyapunov stability has been established.

## 1.3 Qualitative Analysis of Large Scale Systems: General Remarks

Despite its elegance and generality, the usefulness of the Lyapunov approach is severely limited when applied to problems of high dimension and complex interconnecting structure. For this reason it is frequently advantageous to view high order systems as being composed of several lower order subsystems which when interconnected in an appropriate fashion, yield the original *composite* or *interconnected system*. The stability analysis of such systems can often then be accomplished in terms of the simpler subsystems and in terms of the interconnecting structure of such composite systems. In this way, complications which usually arise when the direct method is applied to high order systems may often be avoided. This is precisely the approach which we will employ. Indeed, since statements similar to the above apply equally as well to problems involving Lagrange stability, estimates of trajectory behavior and trajectory bounds, input–output stability, and the like, throughout this book we will pursue a method of qualitative analysis of large systems at different hierarchical levels. As will be pointed out repeatedly, in a certain sense, the method of analysis advanced in this book should be viewed as being more important than the individual results presented.

There is a sizable body of literature concerned with various aspects of qualitative analysis of large dynamical systems. Since this subject is rather new, and since the literature is still growing, it is difficult and quite pointless to make an attempt at citing every reference. Indeed, if we were to list all sources concerned with this topic, such a list would be too long and not justified. On the other hand, if we were to confine the listings only to immediate references used, such a list would be too short and would hardly reflect the importance of this subject. For this reason we shall compromise and mention only some selected results which are in the spirit of the method of analysis proposed herein.

1. *Systems Described by Ordinary Differential Equations* (Lyapunov Stability). Using *vector Lyapunov functions*, originally introduced by Bellman [2], Bailey [1, 2] invoked the comparison principle (see, e.g., Walter [1]) to establish sufficient conditions for exponential stability of interconnected systems described by nonlinear nonautonomous ordinary differential equations with exponentially stable subsystems and with linear time-invariant interconnecting structure. Subsequently, results for exponential stability, uniform asymptotic stability, instability, and complete instability of composite systems represented by nonlinear nonautonomous ordinary differential equations with nonlinear and time varying interconnections and with subsystems that may be exponentially stable, uniformly asymptotically stable (and sometimes unstable) were obtained by Piontkovskii and Rutkovskaya [1], Thompson [1, 2], Porter and Michel [1, 2], Michel and Porter [3], Thompson and Koenig [1], Matrosov [1], Araki [1, 4], Araki and Kondo [1], Michel [5–7, 9], Matzer [1], Grujić and Siljak [2], Weissenberger [1], Bose [1], Bose and Michel [1, 2], and others. In some of these results, vector Lyapunov functions as well as scalar Lyapunov functions consisting of a weighted sum of Lyapunov functions for the free or isolated subsystems are employed. In addition, absolute stability results for interconnected systems endowed with several nonlinearities were established by several authors (see, e.g., McClamroch and Ianculescu [1], Bose [1], Bose and Michel [1, 2], and Blight and McClamroch [1]). Additional related Lyapunov stability results for large systems are contained in the paper by Tokumaru, Adachi, and Amemiya [2] and in the survey paper by Athans, Sandell, and Varaiya [1].

2. *Systems Described by Difference Equations and Sampled Data Systems* (Lyapunov Stability). Sufficient conditions for uniform asymptotic stability, exponential stability, and instability of composite systems described by nonautonomous nonlinear difference equations were established by Araki, Ando, and Kondo [1], Michel [5, 7], and Grujić and Siljak [1, 3]. Results for uniform asymptotic stability of sampled data composite systems (i.e., hybrid finite-dimensional systems) were obtained by Michel [5, 7].

3. *Systems Described by Stochastic Differential Equations and Stochastic Difference Equations* (Lyapunov Stability). The Lyapunov stability of large scale systems described by stochastic differential equations (Ito equations as well as other types of stochastic differential equations) and stochastic difference equations are treated in papers by Michel [8, 10, 11], Michel and Rasmussen [1–3], Rasmussen and Michel [1, 3], and Rasmussen [1]. Scalar and vector Lyapunov functions are used in the analysis. In addition to Wiener processes, disturbances modeled by Poisson step processes, jump Markov processes, and the like, are considered. The approach in these references is general enough to allow disturbances to enter into the subsystem structure and into the interconnecting structure. Composite systems with stable as well as unstable subsystems are treated. The obtained results yield sufficient conditions for asymptotic stability with probability one and in probability and exponential stability with probability one, in probability and in the quadratic mean.

4. *Infinite-Dimensional Systems* (Lyapunov Stability). Matrosov [2] uses vector Lyapunov functions while Michel [5, 7], Rasmussen and Michel [2, 4], and Rasmussen [1] use scalar Lyapunov functions (consisting of a weighted sum of Lyapunov functions for the free or isolated subsystems) to obtain conditions for uniform asymptotic stability and exponential stability of composite systems described by differential equations which are defined on Hilbert and Banach spaces. These results are general enough to allow analysis of large systems which can be represented by ordinary differential equations, differential difference equations, functional differential equations, integrodifferential equations, certain classes of partial differential equations, as well as other types of evolutionary systems. In addition, these results provide a systematic procedure for the stability analysis of hybrid dynamical systems, i.e., dynamical systems described by a mixture of different types of equations.

5. *Lagrange Stability and Estimates of Trajectory Behavior and Trajectory Bounds*. Sufficient conditions for uniform boundedness and uniform ultimate boundedness of solutions of interconnected systems described by ordinary differential equations were obtained by Bose and Michel [1, 2] and Bose [1]. Estimates of trajectory behavior and trajectory bounds of composite systems described by ordinary differential equations were established in references by Michel [3, 4, 9], Matrosov [1, 2], and Michel and Porter [1, 2]. These results include earlier ones for finite time stability of systems described on product spaces (see Weiss and Infante [1]).

6. *Input–Output Stability*. Conditions for the input–output stability (input–output boundedness and continuity) for large classes of interconnected dynamical systems are established in Porter [1], Porter and Michel [3–5], Tokumaru, Adachi, and Amemiya [1], Lasley and Michel [1–6],