NONLINEAR DIFFERENTIAL EQUATIONS

Invariance, Stability, and Bifurcation

Edited by
PIERO de MOTTONI
and
LUIGI SALVADORI

NONLINEAR DIFFERENTIAL EQUATIONS

Invariance, Stability, and Bifurcation

Edited by

PIERO de MOTTONI

Istituto per le Applicazioni
del Calcolo "Mauro Picone"
CNR
Roma, Italy

LUIGI SALVADORI

Dipartimento di Matematica
Libera Università di Trento
Trento, Italy

ACADEMIC PRESS 1981

A Subsidiary of Harcourt Brace Jovanovich, Publishers

New York London Toronto Sydney San Francisco

ACADEMIC PRESS, INC.
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS, INC. (LONDON) LTD.
24/28 Oval Road, London NW1 7DX

Library of Congress Cataloging in Publication Data
Main entry under title:

Nonlinear differential equations.

Proceedings of a conference held in Trento, Italy, May 25-30, 1980.
1. Differential equations, Nonlinear--Congresses.
I. De Mottoni, Piero. II. Salvadori, Luigi.
QA370.N65 515.3'55 81-543
ISBN 0-12-508780-2

PRINTED IN THE UNITED STATES OF AMERICA

81 82 83 84 9 8 7 6 5 4 3 2 1

Contributors

Numbers in parentheses indicate the pages on which the authors' contributions begin.

Norman W. Bazley (1), *Mathematisches Institut, Universität Köln, Weyertal 86-90, D-5000 Köln, West Germany*

Edoardo Beretta (11), *Istituto di Biomatematica, Via A. Saffi, 1, I-61029, Urbino, Italy*

S. R. Bernfeld (29), *Department of Mathematics, The University of Texas at Arlington, Box 19408, Arlington, Texas 76019*

Marco Biroli (41), *Istituto Matematico, Politecnico di Milano, Via Bonardi, 9, I-20133 Milano, Italy*

Victor I. Blagodatskikh (55), *Steklov Mathematical Institute, 117393, Vsvilova 42, Moscow, U.S.S.R.*

Moses Boudourides (59), *Department of Mathematics, Democritus University of Thrace, School of Engineering, Xanthi, Greece*

Vincenzo Capasso (65), *Istituto di Analisi Matematica, Palazzo Ateneo, I-70121 Bari, Italy*

Silvia Caprino (77), *Istituto di Matematica, Università di Camerino, Via V. Venanzi, I-62032, Camerino, Italy*

M. Cecchi (85), *Istituto di Matematica Applicata, Università di Firenze, Via S. Marta, 3, I-50139 Firenze, Italy*

Nathaniel Chafee (97), *School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332*

Klaus Deimling (129), *Gesamthochschule Paderborn, Fachbereich 17, Mathematik-Informatik, Warburgerstrasse 100, D-4790 Paderborn, West Germany*

Odo Diekmann (133), *Mathematisch Centrum, Kruislaan 413, NL-1098 SJ Amsterdam, The Netherlands*

Giorgio Fusco (145), *Istituto di Matematica Applicata, Università di Roma, Via A. Scarpa, 10, I-00161 Roma, Italy*

Stephan van Gils (133), *Mathematisch Centrum, Kruislaan 413, NL-1098 SJ Amsterdam, The Netherlands*

Jesús Hernández (161), *Departamento de Matemática, Universidad Autónoma, Ciudad Universitaria de Cantoblanco, Madrid 34, Spain*

Nicoletta Ianiro (175), *Istituto di Meccanica e Macchine, Universita dell'Aquila, Monteluco-Roio, I-67100 L'Aquila, Italy*

R. Kannan (183), *Department of Mathematics, The University of Texas at Arlington, Box N 19408, Arlington, Texas 76019*

Nicholas D. Kazarinoff (195), *Department of Mathematics, SUNY at Buffalo, 106 Diefendorf Hall, Buffalo, New York 14214*

Hansjörg Kielhöfer (207), *Institut für Angewandte Mathematik, Universität Würzburg, Am Hubland, D-8700 Würzburg, West Germany*

Klaus Kirchgässner (221), *Mathematisches Institut, Universität Stuttgart, Pfaffenwaldring 57, D-7000 Stuttgart, West Germany*

V. Lakshmikantham (243), *Department of Mathematics, The University of Texas at Arlington, Box 19408, Arlington, Texas 76019*

S. Leela (259), *Department of Mathematics, SUNY at Geneseo, Geneseo, NY 14454*

Carlotta Maffei (175),*Istituto di Matematica, Università di Camerino, Via V. Venanzi, I-62032 Camerino, Italy*

M. Marini (85), *Istituto di Matematica Applicata, Università di Firenze, Via S. Marta, 3, I-50139 Firenze, Italy*

Jean Mawhin (269), *Institut Mathématique, Université de Louvain, Chemin du Cyclotron, 2, B-1348 Louvain-la-neuve, Belgium*

Ju. A. Mitropolsky (283), *Institute of Mathematics, Academy of Sciences of the Ukrainian SSR, Repina, 3, 252601 Kiev, U.S.S.R.*

Piero de Mottoni (327), *I.A.C. "M. Picone", C.N.R., Viale del Policlinico, 137, I-00161 Roma, Italy*

P. Negrini (29), *Istituto di Matematica, Università di Camerino, Via V. Venanzi, I-62032 Camerino, Italy*

L. Salvadori (29), *Dipartimento di Matematica, Libera Università di Trento, I-38050 Povo (Trento), Italy*

Andrea Schiaffino (327, 339), *Istituto Matematico "G. Castelnuovo", Universita di Roma, Piazzale Aldo Moro, 5, I-00185 Roma, Italy*

Alberto Tesei* (339), *I.A.C. "M. Picone", C.N.R., Viale del Policlinico, 137, I-00161 Roma, Italy*

Rosanna Villella-Bressan (347), *Istituto di Analisi e Meccanica Università, Via Belzoni, 7, I-35100 Padova, Italy*

P. L. Zezza (85), *Istituto di Matematica, Università di Siena, Via del Capitano, 15, I-53100 Siena, Italy*

**Present address: Istituto di Matematica Applicata, Università di Roma, Via A. Scarpa, 10, I-00161 Roma, Italy*

Preface

An international conference on Nonlinear Differential Equations: Invariance, Stability, and Bifurcation was held at the Villa Madruzzo, Trento, Italy, May 25–30, 1980. The conference is part of a series of meetings sponsored by the Centro Interuniversitario per la Ricerca Matematica (CIRM) and by the Italian Council for Scientific Research (CNR). It is a pleasure to acknowledge the support received from the sponsoring agencies, which made the conference possible.

The purposes of the conference were to highlight developments in the qualitative theory of nonlinear differential equations, and to promote the exchange of mathematical ideas in stability and bifurcation theory. The mutual interaction and cooperation between qualified researchers, active both in theoretical and applied investigations, proved extremely fruitful and stimulating.

The present volume consists of the proceedings of the conference. It includes papers that were delivered as survey talks by Professors N. Chafee, K. Kirchgässner, V. Lakshmikantham, and Ju. A. Mitropolsky, as well as a number of research reports. A number of contributions focus on the interplay between stability exchange for a stationary solution and the appearance of bifurcating periodic orbits. Another group of papers deals with the development of methods for ascertaining boundedness and stability. Nonlinear hyperbolic equations are considered in further contributions, featuring, among others, stability properties of periodic and almost periodic solutions. Papers devoted to the development of bifurcation and stability analysis in nonlinear models of applied sciences are also included.

We wish to express our appreciation to Mr. A. Micheletti, secretary of CIRM, for assisting us in organizing the conference. A special grant of the CNR, which we warmly acknowledge, made the typing of the proceedings possible. This was carefully carried out at the Centro Stampa KLIM, Rome.

Contents

ABSTRACT NONLINEAR WAVE EQUATIONS: EXISTENCE, LINEAR AND MULTI-LINEAR CASES, APPROXIMATION, STABILITY

Norman W. Bazley [1]

Mathematics Institute,
University of Cologne
West Germany

1. INTRODUCTION

We consider the initial value problem for a nonlinear wave equation given by

$$u''(t) + Au(t) + M(u(t)) = 0$$
$$u(0) = \varphi, \quad u'(0) = \psi. \tag{1}$$

Here A is a strictly positive, self-adjoint operator in a separable Hilbert space $\mathcal{H}$, with domain D_A; the initial values satisfy $\varphi \in D_A$, $\psi \in D_{A^{\frac{1}{2}}}$.

The purpose of this article is to survey some recent existence and approximation results for (1). Such problems were first studied by K. Jörgens [9] and I. Segal [14]. In 1970 F. Browder [4] carried through an operator theoretic study of the above equation, which was recently simplified and generalized by E. Heinz and W. von Wahl [8]. These results were in-

[1] Supported in part by the European Research Office of the U.S. Army.

ISBN 0-12-508780-2

vestigated in the Diplomarbeit of D. Kremer [11], which motivated other Diplomarbeiten [5,15] and studies [1-3] at the University of Cologne. Related works appear in [6,7,12].

2. EXISTENCE THEORY OF BROWDER - HEINZ - von WAHL

In [8] the following two assumptions, introduced in [4], are used to prove the existence of a strong solution of (1):

$$\underline{\text{Assumption}}: \quad M: D_{A^{\frac{1}{2}}} \to \mathcal{H} \text{ is a mapping defined on all of } D_{A^{\frac{1}{2}}}; \tag{2}$$

<u>Assumption A</u>: For every $c > 0$ there exists a $\kappa(c)$ such that

$$\|M(A^{-\frac{1}{2}}u) - M(A^{-\frac{1}{2}}v)\| \leq \kappa(c)\|u - v\| \text{ for } \|u\|, \|v\| \leq c. \tag{3}$$

The first assumption is often easy to verify in applications, even though $A^{\frac{1}{2}}$ itself is usually not known in closed form. However, the assumption (3) that $M(A^{-\frac{1}{2}}u)$ satisfies a Lipschitz condition on spheres is usually more difficult to check. We refer to it as "Assumption A of Browder-Heinz-von Wahl".

When both assumptions (2) and (3) are satisfied we have by [8]:

<u>Theorem</u>. *For arbitrary vectors* $\varphi \in D_A$ *and* $\psi \in D_{A^{\frac{1}{2}}}$ *there exists a unique solution* $u = u(t;\varphi,\psi)$ *of* (1) *in an interval* $0 \leq t < \tilde{T}$, *and* $u(t)$ *belongs to the class* $C^2([0,\tilde{T}),\mathcal{H}) \cap C^1([0,\tilde{T}), D_{A^{\frac{1}{2}}}) \cap C^0([0,\tilde{T}),D_A)$. *If* $\tilde{T} < \infty$, $\lim_{t\uparrow\tilde{T}} E(t) = +\infty$, *where* $E(t) = \|A^{\frac{1}{2}}u(t)\|^2 + \|u'(t)\|^2$.

Thus there exists a unique C^2 solution $u(t)$, which has the representation $u(t) = \sum_{i=1}^{\infty} a_i(t)u_i$, for $\{u_i\}_1^{\infty}$ a complete orthonormal system in $\mathcal{H}$.

We outline some of the steps used in proving the above theorem. For details, the reader is referred to [8]. There Heinz and von Wahl consider the Banach space $X = C^0([0,T],D_{A^{\frac{1}{2}}})$

with norm $\|u\|_X = \max_{0\le t\le T} \|A^{\frac{1}{2}}u(t)\|$, and for $u \in X$, $n = 1,2,\ldots,$ they define mappings S_n as

$$u \to S_n(u) = w(t) - \int_0^t \sin(t-s)A^{\frac{1}{2}}(A^{-\frac{1}{2}}M_n(u(s)))ds, \tag{4}$$

where

$$w(t) = \cos tA^{\frac{1}{2}}\varphi + \sin tA^{\frac{1}{2}}(A^{-\frac{1}{2}}\psi).$$

Here, for $u \in D_{A^{\frac{1}{2}}}$, they define $M_n(u) = M(E_n u)$, where E_λ is the resolution of the unity for A. From Assumption A, each S_n has a fixed point $v_n(t)$ with $\|v_n\|_X \le c$, where $c = \|A^{\frac{1}{2}}\varphi\| + \|\psi\| + 1$, and $0 \le t \le T = 1/(2\kappa(c))$. Further, detailed arguments show that the $v_n(t)$ converge to a unique strong solution of (1) in the norm

$$\lim_{n\to\infty} \sup_{0\le t\le T} E(u(t) - v_n(t)) = 0 \tag{5}$$

where

$$E(u(t)) = \|A^{\frac{1}{2}}u(t)\| + \|u'(t)\|. \tag{6}$$

The same arguments are re-applied at the endpoint T, to obtain a strong solution in a larger time interval. Extension to the maximal interval $0 \le t < \tilde{T}$ is obtained by repeated applications.

3. ASSUMPTION A IN SIMPLE CASES

In this section we present some recent results of K.-G. Strack and the author in some previously overlooked simple cases. Proofs and applications are given in [3].

We first consider the special case of (1) where $M(u) = Bu$, for B an unbounded, linear operator with $D_B \supseteq D_{A^{\frac{1}{2}}}$. We remark that if A is an elliptic differential operator of order 2p, then $D_{A^{\frac{1}{2}}} = \overset{\circ}{H}{}^p$ and the domain condition (2) only allows differential operators of order p or less, in agreement with Goldstein [7]. For this case Assumption A becomes $\|BA^{-\frac{1}{2}}u\| \le \kappa\|u\|$, which is equivalent to $\|A^{\frac{1}{2}}x\|^2 \ge \varepsilon\|Bx\|^2$, for all $x \in D_{A^{\frac{1}{2}}}$, where $x = A^{-\frac{1}{2}}u$ and $\varepsilon = 1/\kappa^2$. Thus we have

Lemma. Assumption A is satisfied if and only if there is $\varepsilon > 0$ such that $(A^{\frac{1}{2}}x, A^{\frac{1}{2}}x) - \varepsilon(Bx, Bx) \ge 0$ for all x in $D_{A^{\frac{1}{2}}}$. Well known results of Kato [10] lead to the alternative formulation:

Lemma. Let $D_{B^*B} \supseteq D_A$ and suppose there exists a $\delta > 0$ such that $\|B^*Bu\| \le \delta\|Au\|$ for all u in D_A. Then Assumption A is satisfied if and only if there exists $\varepsilon > 0$ such that $A - \varepsilon B^*B \ge 0$ on D_A.

Similar results hold if M(u) has a Gâteaux derivative M_u' for every u in $D_{A^{\frac{1}{2}}}$ and $\sup\{\|M_v'A^{-\frac{1}{2}}\|;\ v \in D_{A^{\frac{1}{2}}}\} < \infty$. Then we have [16] the global Lipschitz condition

$$\|M(A^{-\frac{1}{2}}u) - M(A^{-\frac{1}{2}}v)\| \le \Big(\sup_{v \in D_{A^{\frac{1}{2}}}} \|M_v'A^{-\frac{1}{2}}\|\Big)\cdot\|u - v\|. \tag{7}$$

One such result reads:

Lemma. Let $\varepsilon > 0$ be such that $A - \varepsilon M_v'^{*}M_v'$ is essentially self-adjoint on each $D_A \cap D_{M_v'^*M_v'}$. Further, let $0 < \varepsilon_0 \le \varepsilon$ be such that $A - \varepsilon_0 M_v'^*M_v' \ge 0$ for all $v \in D_{A^{\frac{1}{2}}}$. Then Assumption A holds.

Many nonlinearities arising in mathematical physics are generated by α-linear forms, where α is a positive integer. That is, there exists an α-linear form $\tilde{M}(u_1,\ldots,u_\alpha)$ such that $M(u) = \tilde{M}(u,\ldots,u)$. For this case one can easily show:

Lemma. Let $M(A^{-\frac{1}{2}}u_1,\ldots,A^{-\frac{1}{2}}u_\alpha)$ be a bounded α-linear form. Then Assumption A holds.

4. FAEDO-GALERKIN APPROXIMATIONS

In [1,2] we considered the approximation of a solution $u(t) = \sum_{i=1}^{\infty} \alpha_i(t)u_i$ of (1). Our idea was to consider the time dependent infinite vectors $\vec{\alpha}(t)$ with components $\{\alpha_1(t), \alpha_2(t), \ldots\}$ in the Hilbert space ℓ^2. We further made the

Assumption B: The operator A has a pure point spectrum, $Au_i = \lambda_i u_i$, $i = 1,2,\ldots$, with a complete orthonormal system of eigenvectors $\{u_i\}_1^{\infty}$.

Then we approximated the first n components of $\vec{\alpha}(t)$ by n-dimensional components $\vec{\alpha}^n(t) = \{\alpha_1^n(t), \ldots \alpha_n^n(t)\}$ of the Faedo-Galerkin approximations

$$P^n v = \sum_{i=1}^{n} \alpha_i^n(t) u_i. \tag{8}$$

Here $P^n v$ satisfies the equations

$$\begin{gathered} \frac{d^2}{dt^2} P^n v + P^n A P^n v + P^n M(P^n v) = 0, \\ P^n v(0) = P^n \varphi, P^n v'(0) = P^n \psi. \end{gathered} \tag{9}$$

The nonlinear term $M(P^n v)$ has the form

$$M(P^n v) = M(\sum_{j=1}^{n} \alpha_j^n u_j) = \sum_{i=1}^{m(n)} \beta_i(\alpha_1^n, \ldots \alpha_n^n) u_i, \tag{10}$$

where

$$\beta_i = (M(\sum_{j=1}^{n} \alpha_j^n u_j), u_i), \quad i = 1, \ldots m(n), \tag{11}$$

and m(n) is either finite or infinite. Then the equations (9) reduce to the system of second order ordinary differential equations

$$\begin{gathered} \ddot{\alpha}_i^n + \lambda_i \alpha_i^n + \beta_i(\alpha_1^n, \ldots \alpha_n^n) = 0 \\ \alpha_i^n(0) = (\varphi, u_i), \ \dot{\alpha}_i^n(0) = (\psi, u_i), \ i = 1, \ldots n. \end{gathered} \tag{12}$$

Our principal result is that the Faedo-Galerkin approximations $P^n v$ are identical with $P^n v_n$, the projections of the fixed points v_n of the operators S_n in (4). This follows since $E_{\lambda_n} v = P^n v$ implies that v_n satisfies

$$\frac{d^2 v_n}{dt^2} + Av_n + M(P^n v_n) = 0$$
$$v_n(0) = \varphi, \quad v_n'(0) = \psi. \tag{13}$$

Operating on (13) by P^n and noting that $P^n A = P^n A P^n$ leads to (9). We can thus carry over the convergence results (5) of Browder-Heinz-von Wahl to the estimation of $\{\alpha_1(t), \alpha_2(t) \dots\}$ by $\{\alpha_1^n(t), \dots, \alpha_n^n(t)\}$, the solutions of (12). For this purpose we introduce the norms E_n defined by

$$E_n(\vec{\alpha}(t) - \vec{\alpha}^n(t)) = \{\sum_{i=1}^{n} \lambda_i (\alpha_i(t) - \alpha_i^n(t))^2\}^{\frac{1}{2}}$$
$$+ \{\sum_{i=1}^{n} (\alpha_i'(t) - \alpha_i^{n\prime}(t))^2\}^{\frac{1}{2}}, \tag{14}$$

and set $E_\infty(\vec{\alpha}) = E(\vec{\alpha}) = E(u(t))$. Detailed arguments [2] show that convergence in each compact interval holds according to the following

<u>Theorem</u>. For each $\hat{T} < \tilde{T}$ we have

$$\lim_{n \to \infty} \sup_{0 \le t \le \hat{T}} E_n(\vec{\alpha}(t) - \vec{\alpha}^n(t)) = 0. \tag{15}$$

It follows that

$$E(\vec{\alpha}(t)) = \lim_{n \to \infty} E_n(\vec{\alpha}^n(t)), \tag{16}$$

for $0 \le t \le \hat{T}$. Thus $E_n(\vec{\alpha}^n(t))$, the energy of the approximating systems (12), and $E(\vec{\alpha}(t))$, the energy of the solution (1), blow up and remain small together. In particular, for $\varepsilon > 0$ we have $E(\vec{\alpha}(t)) < \varepsilon$, $0 \le t < \infty$, if and only if $\lim_{n \to \infty} E_n(\alpha^n(t)) < \varepsilon$, $0 \le t < \infty$, a fact which will be used in the next section on stability analysis.

Finally, we make two remarks on the above procedure. Firstly, in a qualitative or quantitative analysis of (1) by the system (12) it is extremely useful to have an explicit expression for the functions $\beta_i(\alpha_1^n,\ldots,\alpha_n^n)$ in (11); if so, we say [1] that the nonlinearity $M(u)$ is "reproducing" relative to the sequence $\{u_i\}_i^\infty$. The idea is that many nonlinearities have computable expansion coefficients, when applied to a suitable complete orthonormal system. An explicit example is given in the next section.

The second remark concerns the case when the nonlinearity $M(u)$ is the gradient of a functional $\Phi(u)$; that is, $\Phi(u+h) - \Phi(u) = (M(u),h) + R_u(h)$, where $R_u(h)$ denotes terms of higher order in h. This holds, for example, when $M(u)$ is a cyclically monotone operator. Then the approcimations (12) have the form of a conservative Hamiltonian system of classical mechanics, and

$$\frac{\partial\beta_i}{\partial\alpha_j^n} = \frac{\partial\beta_j}{\partial\alpha_i^n} \qquad 1 \leq i,\ j \leq n.$$

Thus there exists a potential $V_n(\alpha_1^n,\ldots,\alpha_n^n)$ and we can write (12) in the form

$$\begin{aligned} &\ddot{\vec{\alpha}}_n + \operatorname{grad} V_n(\alpha_1^n,\ldots,\alpha_n^n) = 0 \\ &\alpha_i^n(0) = (\varphi,u_i),\ \dot{\alpha}_i^n(0) = (\psi,u_i),\ i = 1,\ldots,n. \end{aligned} \tag{17}$$

5. STABILITY

We announce some new results of K.-G. Strack, who in [15] obtained qualitative results for (1) in special cases. In particular, he considered $u(x,t)$ in $\mathcal{H} = L^2(0,1)$ as a solution of the equation

$$\begin{aligned} &u_{tt} - u_{xx} + u^3 = 0 \\ &u(0,t) = u(1,t) = 0 \\ &u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x). \end{aligned} \tag{18}$$

Here $Au = -u''$, $u(0) = u(1) = 0$, which has a point spectrum diverging to infinity. The eigenfunctions are given by $u_i = \sqrt{2} \sin i\pi x$, $i = 1,2,..,$ with corresponding eigenvalues $\lambda_i = i^2\pi^2$, $i = 1,2,...$ The nonlinearity $M(u) = u^3$ is defined on $D_{A^{1/2}}$ and satisfies Assumption A, so that the theorem of Browder-Heinz-von Wahl guarantees the existence of a strong solution $u(t)$.

Strack investigates the stability of the zero solution $u(x,t) = 0$ of (18), that is, $\varphi(x) = \psi(x) = 0$. His analysis makes essential use of the results of Rutkowski [13], who use the reproducing property of the nonlinearity to obtain explicit expansion coefficients $\beta_i(\alpha_1^n,...,\alpha_n^n)$ for (10). This allows the construction of Lyapunov functions for the n^{th} approximation (12). Detailed arguments [15] lead to the following stability theorem for (18).

Theorem. Let $\varepsilon > 0$. There exists a δ, independent of n, such that $\|A\varphi\|_2 + \|\psi\|_2 < \delta$ implies $E_n(\vec{\alpha}^n) < \varepsilon$ for all $n \in \mathbb{N}$, $t \geq 0$.

By the convergence result in (16) we have that the zero solution is stable in the energy norm.

REFERENCES

1. N.W. Bazley: Approximation of Wave Equations with Reproducing Nonlinearities, Nonlinear Analysis TMA 3, 539-546, (1979).
2. N.W. Bazley: Global Convergence of Faedo-Galerkin Approximations to Nonlinear Wave Equations, Nonlinear Analysis TMA 4, 503-507, (1980).
3. N.W. Bazley, K.-G. Strack: Self-adjoint Sums of Operators and the Theory of Nonlinear Wave Equations, Univ. Cologne Technical Report, 1980.
4. F. Browder: Existence Theorems for Nonlinear Partial Differential Equations. In Global Analysis, Proc. Symp. Pure Math. vol. 16, Providence, AMS, 1970, 1-60.
5. L. Brüll: Diplomarbeit, Universität zu Köln, 1980.