THE SMITH CONJECTURE

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and
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PREFACE

In the 1940s Paul A. Smith asked whether or not the fixed point set of a periodic, orientation-preserving homeomorphism of S^3 to itself was always an unknotted circle. In the fall of 1978 this question was answered in the affirmative for diffeomorphisms. The proof rests on the work of mathematicians in diverse areas of mathematics. In the spring of 1979 a symposium was held at Columbia University on the solution to Smith's question. It brought together the principal actors in the drama to present various pieces of the proof. In addition to written versions of the presentations, this volume includes an introduction which explains how the pieces fit together. (See especially Chapter III, Section 3.) There are also two papers (Chapters IX and X) containing generalizations of the results on the Smith conjecture. The last article (Chapter XI) is a discussion of the situation in dimensions greater than three.

It seemed entirely appropriate to have such a symposium at Columbia University. Paul Smith spent most of his mathematical life at Columbia. In the spring of 1979 he was Professor Emeritus at Columbia and still lived in the neighborhood. He was one of the most attentive members of the audience as the resolution of his 38-year-old question unfolded.

It was also at Columbia that a significant step in the solution of the conjecture occurred. During a conversation with Bass, Thurston learned of Bass's result (Chapter VI). He saw, in the light of his own work (Chapter V), the relevance to the Smith conjecture. He also saw the need to treat the cases covered in Part C. What was needed to deal with these missing cases came clearly into focus during conversations between Thurston and Gordon (the latter being motivated by his earlier work with Litherland; see Chapter VII). At about the same time, Meeks and Yau had established exactly the required result (Chapter VIII). However, there was a gap in communication between Thurston and Gordon, on the one hand, and Meeks and Yau, on the other. This gap was bridged by Gordon when he learned of the existence of the work of Meeks and Yau. With that, the proof was complete.

There was a purpose beyond the purely mathematical in holding the symposium. That was to honor Paul Smith. His work has had tremendous influence on topology, and the symposium provided a look at one direction in which this influence has led the field. During the symposium Smith said that out of consideration for the younger mathematicians he would be sure to make his next question easier to solve than this one. He was jesting of course, for he knew full well that mathematics needs deep and hard problems and that the younger mathematicians assembled at the symposium owed him a debt of gratitude for making his questions hard and fertile ones on which to work.

Sadly, nothing marks the passage of time between the symposium and the publication of this volume more clearly than Paul Smith's death. All who knew him are saddened and made poorer by his passing. He was a fine man as well as a first-rate mathematician.

ACKNOWLEDGMENTS

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LIST OF NOTATION

$\tilde{\Sigma}$	Homotopy 3-sphere
$h\colon \tilde{\Sigma} \to \tilde{\Sigma}$	Periodic diffeomorphism
Σ	The quotient space of $\tilde{\Sigma}$ by the action generated by h
$\tilde{K}$	The fixed set of h
$p\colon \tilde{\Sigma} \to \Sigma$	Natural projection
$K = p(\tilde{K})$	
$\Gamma = \pi_1(\Sigma - K)$	
$\tilde{\Gamma} = \pi_1(\tilde{\Sigma} - \tilde{K})$	
$M = \Sigma - K$	
$F \subset M$	Closed surface
$\tilde{F} = p^{-1}(F)$	
D	2-Disk
N	A closed tubular neighborhood of K
$\tilde{N} = (p^{-1}(N))$	
μ	Meridian on N
B	3-Ball

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PART A. INTRODUCTION

PART A

INTRODUCTION

CHAPTER I

The Smith Conjecture

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1. Formulations

Let S^3 be the unit sphere in $\mathbf{R}^4$. Smith proved [Sm 1] that any periodic, orientation-preserving homeomorphism of S^3 to itself with fixed points has a fixed point set that is homeomorphic to a circle. He then asked [Ei 1] if the fixed point set must be unknotted. (Unknotted here means that there is a homeomorphism of S^3 to itself that throws the given simple closed curve onto a geometric circle, i.e., onto S^3 intersected with a two-dimensional linear subspace of $\mathbf{R}^4$.) As Smith realized, an affirmative answer to this question is equivalent to the statement that every such homeomorphism of S^3 to itself is standard, that is, topologically equivalent to a linear (i.e., orthogonal) action [Moi 1, Sm 4]. Montgomery and Zippin [Mon–Z] showed that in this generality the answer to the question is no. They gave examples of periodic homeomorphisms of S^3 whose fixed point sets are wildly embedded

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simple closed curves (wild in the sense that there are not even *local* homeomorphisms that throw the fixed point set onto a standard arc). This type of pathology is ruled out if we require the homeomorphism to be a diffeomorphism (or piecewise linear (PL) homeomorphism). One formulates the differentiable version of Smith's question thus:

If $h: S^3 \to S^3$ is an orientation-preserving, periodic diffeomorphism with nonempty fixed point set, then is this fixed set an unknotted circle?

What became known as the Smith conjecture was the conjecture that the answer to this question is yes. Another way to frame the question is to ask

Is every orientation-preserving, periodic diffeomorphism $h: S^3 \to S^3$, with fixed points, conjugate (by a diffeomorphism of S^3 to itself) to an orthogonal diffeomorphism?

It is the purpose of this volume to present the recent arguments that answer this question affirmatively.

It turns out that one makes no essential use of the fact that the space being acted upon is S^3. The arguments apply more generally to homotopy 3-spheres. *Henceforth, Σ denotes a homotopy 3-sphere.* The main theorem proved in this volume is the following:

THEOREM (Solution of the Smith Conjecture). *Let $h: \Sigma \to \Sigma$ be an orientation-preserving, periodic diffeomorphism (different from the identity) with fixed points. Then the fixed point set of h is an unknotted circle*[1] *in Σ.*

Remarks. (1) The arguments proving the Smith conjecture can easily be adapted for the PL case, or for the topological case, provided that in the topological case one assumes that the fixed point set $K \subset \Sigma$ is locally flat.

(2) All known examples in which the fixed point set K is wild have the property that K bounds a topologically embedded disk.

There is a reformulation of this result in terms of group actions. Suppose that we are given an effective, orthogonal action of a finite group $G \times S^3 \to S^3$ and a homotopy 3-cell H. Choose a ball $B \subset S^3$ that is disjoint from all its translates under nontrivial elements of G. Remove B and all its translates from S^3 and sew a copy of H into each hole. The restricted G-action on $S^3 - (\bigcup_{g \in G} gB)$ extends in an obvious way to an action on the resulting homotopy 3-sphere. Any action constructed in this manner is called *essentially linear.*

[1] An unknotted circle is one that bounds a differentiably embedded 2-disk $D \subset \Sigma$. If $\Sigma = S^3$, then this notion agrees with the other definition of unknotted.

THEOREM (Reformulation of the Solution to the Smith Conjecture). *Let $G \times \tilde{\Sigma} \to \tilde{\Sigma}$ be a finite cyclic group action generated by an orientation-preserving diffeomorphism with a nonempty fixed point set. This action is equivariantly diffeomorphic to an essentially linear action.*

2. Generalizations

The techniques used to establish the Smith conjecture can be used to prove various generalizations. Two such generalizations are treated in Chapters IX and X of this volume. In Chapter IX Meeks and Yau prove the following:

THEOREM *If G is a finite group of orientation-preserving diffeomorphisms of $\mathbf{R}^3$, then the action is equivariantly diffeomorphic to a linear action, except possibly when G is isomorphic to A_5, the alternating group on 5 letters.*[2]

In Chapter X Davis and Morgan treat another generalization:

THEOREM. *If G is a finite group of orientation-preserving diffeomorphisms of $\tilde{\Sigma}$ so that all isotropy groups are cyclic and one isotropy group has order divisible by a prime > 5, then G is equivariantly diffeomorphic to an essentially linear action.*[3]

These results lead one to the following:

QUESTION. *Is every nonfree action of a finite group on a homotopy 3-sphere or a contractible 3-manifold essentially linear?*

The results quoted above show that in certain special cases the answer is yes. There is no known example of a nonlinear, differentiable action of a finite group on S^3 or $\mathbf{R}^3$.

3. Some Consequences Relating to the Poincaré Conjecture

The theorem on the solution of the Smith conjecture affirms several special cases of the Poincaré conjecture: Suppose that $h: \tilde{\Sigma} \to \tilde{\Sigma}$ is an orientation-preserving periodic diffeomorphism of period n with fixed points. Let Σ be the quotient of $\tilde{\Sigma}$ by the cyclic group generated by h. It is easy to see that Σ is a homotopy 3-sphere. It follows from the theorem on the reformulation

[2] See footnote 1 in Chapter X for the case $G \approx A_5$.

[3] This result has been expanded to cover more cases (see Feighn [Fe]).

of the solution of the Smith conjecture that $\tilde{\Sigma}$ is diffeomorphic to the connected sum of n copies of Σ. In particular, $\tilde{\Sigma}$ is not an irreducible homotopy 3-sphere. Also, if the quotient space Σ is diffeomorphic to S^3, then so is $\tilde{\Sigma}$. This last result can be phrased another way: A cyclic covering of S^3 branched along a smooth knot is never a counterexample to the Poincaré conjecture. On the other hand, not all 3-manifolds (or even all homology 3-spheres) are cyclic branched covers of S^3 [My].

Birman and Hilden [Bir–H] have shown that every 3-manifold with a Heegaard splitting of genus 2 is a two-sheeted branched cyclic cover of S^3. As a result, there is no counterexample to the Poincaré conjecture with a Heegaard splitting of genus 2.

4. Additional Remarks

The proof of the Smith conjecture represents a culmination of the efforts of many mathematicians. Of course, the work of Smith himself on cyclic group actions was seminal. Almost all of the apparatus of three-dimensional topology, as it has developed from the foundation laid by Kneser, Papakyriakopoulos, and Haken, to its contemporary form in the work of Waldhausen, Stallings, Epstein, Jaco, Shalen, and Johannson, among others, is needed as well. In this volume, we regard this apparatus as "classical," as forming the mathematical environment in which the proof resides. The broad outlines of this proof were first brought into focus by Thurston. The proof, as it finally emerged, represents a confluence of ideas and methods from diverse areas of mathematics (minimal surface theory, hyperbolic geometry, and kleinian groups, and the algebra of 2×2 matrices). The applications of these ideas and methods to three-dimensional topology are due to Thurston, Meeks, and Yau, with help from Bass, Shalen, Gordon, and Litherland. The most surprising feature is how well these new techniques mesh with the purely topological techniques already available. Together, they should have a profound influence on three-dimensional topology, the solution of the Smith conjecture being but a beginning.

CHAPTER II

History of the Smith Conjecture and Early Progress

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1. History of the Smith Conjecture

The study of periodic diffeomorphisms of the disk and the sphere began with the work of Brouwer [Bro] and Kerekjarto [Ke] in the 1920s. They proved that an orientation-preserving periodic diffeomorphism of the 2-disk or the 2-sphere was conjugate by a diffeomorphism to a rotation. These original proofs were incomplete and the gap was filled later by Eilenberg [Ei 2].

It was in this context that Smith's work was done. In a series of papers produced during the 1930s and 1940s [Sm 1–3] he studied homeomorphisms of the n-disk and the n-sphere periodic of prime power period p^r. He showed that the $\mathbf{Z}/p$-homology of the fixed point set is the same as that of a smaller dimensional disk or sphere. Furthermore, if the homeomorphism is orientation-preserving, then the codimension of the fixed point set is even. Now consider $f: S^3 \to S^3$ a periodic homeomorphism that preserves the orientation

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