B. S. Razumikhin

Physical Models and Equilibrium Methods in Programming and Economics

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B. S. Razumikhin
Institute for Systems Studies, Academy of Sciences, Moscow

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(Revised and augmented compared to the original Russian edition and the French translation)

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EDITOR'S PREFACE

Approach your problems from the right end and begin with the answers. Then one day, *perhaps you will find the* final question.

'The Hermit Clad in Crane Feathers' in R. van Gulik's The Chinese Maze Murders.

It isn't that they can't see the solution.
It is that they can't see the *problem.*

G. K. Chesterton. The Scandal of Father Brown 'The Point of a Pin'.

Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the "tree" of knowledge of mathematics and related field does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related.

Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces.

This program, Mathematics and Its Applications, is devoted to such (new) interrelations as exempla gratia:

- a central concept which plays an important role in several different mathematical and/or scientific specialized areas;
- new applications of the results and ideas from one area of scientific edeavor into another;
- influences which the results, problems and concepts of one field of enquiry have and have had on the development of another.

The Mathematics and Its Applications programme tries to make available a careful selection of books which fit the philosophy outlined above. With such books, which are stimulating rather than definitive, intriguing rather than encyclopaedic, we hope to contribute something towards better communication among the practitioners in diversified fields.

Because of the wealth of scholarly research being undertaken in the Soviet Union, Eastern Europe, and Japan, it was decided to devote special attention to work emanating from these particular regions. Thus it was decided to start three regional series under the umbrella of the main MIA programme.

The present book is concerned with the mathematics central to the decision sciences, i.e. linear and nonlinear programming, economic equilibrium and growth problems, and optimal control; from an unusual interdisciplinary point of view, however. Namely these matters are studied by means of gas- and fluid-mechanical models and this means that the principles of analytical dynamics and thermodynamics can be applied. This turns out to be quite remarkably fruitful both in terms of suggestion of algorithms (e.g. decomposition algorithms) and in terms of conceptual understanding. Thus e.g. duality in the sense of linear programming turns out to be duality between intensive and extensive variables in thermodynamics, the principle of virtual displacements of analytical mechanics is by and large the same as the Kuhn-Tucker theorem and it turns out that there is some sort of economic potential which is minimal at equilibrium. Also this setting gives a natural and suggestive interpretation to all kinds of penalty function ideas.

It seems to me that the ideas in this book have so far been exploited in only a modest way and that much more can be derived from them (notably the penalty function ones and the economic potential one).

The unreasonable effectiveness of mathematics in science ...

Eugene Wigner

Well, if you knows of a better 'ole, go to it.

Bruce Bairnsfather

What is now proved was once only imagined.

William Blake

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited.

But when these sciences joined company they drew from each other fresh vitality and thenceforward marched on at a rapid pace towards perfection.

Joseph Louis Lagrange

Bussum
April, 1983

Michiel Hazewinkel

PREFACE TO THE REVISED AND AUGMENTED ENGLISH EDITION

The English edition of this book is the result of substantial revisions of the Russian and French editions. The author has not only used the opportunity to remove defects from these editions, he has also introduced essential changes and additions. The most important of these revisions consists of the replacing of chapter IV (finite methods) of the Russian edition with three new chapters: "Principle of removing constraints" (Ch. IV), "The Hodograph method" (Ch. V) and "Method of displacement of elastic constraints" (Ch. VI). In chapter II there is a new section 2.7 "Models for transport problems" because the mechanical and physical models for such problems are elegant, diverse and, especially yield simple constructive devices for solving eigenvalue problems.

The book is devoted to analogies which have always played and will always play a most important role in the never ending progress of science. It seems, therefore, fitting to conclude this short preface with the words of Johannes Kepler: "And above all I value Analogies, my most faithful readers. They pertain to all secrets of nature and can be least of all neglected."

PREFACE TO THE ORIGINAL RUSSIAN EDITION

The first goal of this book is to extend the principles and methods of analytical mechanics and thermodynamics to mathematical programming and mathematical economics. To understand the results obtained, one does not need to be a sophisticated mathematician; the models from physics and mechanics which are used are simple and, in fact, are restricted to models from continuum mechanics dealing with imcompressible liquids and perfect gases. Each model represents a primal and dual problem at the same time and thus, the essential results of linear and nonlinear programming and mathematical economics acquire a physical interpretation (with the difference that in the first case one has extrinsic variables as state parameters and in the second case, intrinsic ones). The equilibrium state of the model thus defines the optimal vectors for the two associated problems and between these vectors there are simple relations resulting from the Clapeyron-Mendeleev equation. Similar results hold for models from mathematical economics which are interpreted in terms of quasi- or pseudo-static transformations of physical systems.

A substantial number of pages are devoted to numerical methods for solving mathematical programming and mathematical economics problems. These derive their mathematical description from the transition processes in the (corresponding) physical systems from an arbitrary initial state to an equilibrium state. The central idea in this part of the book is to control these processes by means of a series of temporary and time-independent constraints which permit the decomposition of the (spontaneous) time evolution towards an unknown equilibrium into several elementary processes for which a mathematical description presents no difficulties. The convergence of a corresponding algorithm then follows from the general principles of thermodynamics.

The value of ideas, concepts, and principles is measured in terms of the number of scientific disciplines in which they find useful applications. The author will have reached his goals if he succeeds in showing that the principles of analytical mechanics and thermodynamics can be effectively applied to economic research.

It is also necessary to make clear what the author understands by the words "equilibrium theory methods". The book does not pretend to expound an economic theory. Its aim is more modest: to show that certain economical-mathematical models (mostly wellknown) lead to equilibrium problems which are analogous to those which come from analytical mechanics and from thermodynamics. Equilibrium theory is a system of principles, methods, and concepts which are the basis of statics, thermodynamics, and the theory of stability.

From Archimedes and Galileo to Lyapunov and Gibbs, this theory of equilibrium has evolved from a search for conditions which equilibrium states must satisfy to the study of conditions under which a given process, that is movement, is possible. The fundamental idea is the existence of a state function the maxima and minima of which correspond to real stable equilibria or real stable movements of the system. This idea has led, in the first case, to the theory of potentials (Leibnitz, Lagrange, Helmholz, Clausius, Gibbs) and, in the second case, to the variational principle of dynamics (Maupertuis, Euler, Lagrange, Hamilton, Jacobi, Poincaré). The ideas, concepts, and methods of equilibrium theory also apply to dynamical problems which formally reduce to problems of statics. To see this, it suffices to recall the principle of d'Alembert, the variational equations of Poincaré, and the stability theory of Lyapunov.

It is possible that a number of mathematicians will find that there is too much mechanics and physics in the book. Would it not have been much simpler to present the results of this book in the language of pure mathematics while pointing out in various sections (which, for that matter, can be skipped in reading the book) the possible mechanical and thermodynamical analogues? It is even possible that, in that case, the book would have found more readers, but the author would have been less happy with his work. The time is no more that any mortal can oversee all or a great part of science. Indeed, "The consequence of the vast and rapidly growing extent of our positive knowledge was a division of labour in science right down to the minutest detail, almost reminiscent of a modern factory where one person does nothing but measure carbon filaments, while another cuts them, a third welds them in and so on. Such a division of labour certainly helps greatly to promote rapid progress in science and is indeed indispensable for it; but just as certainly it harbours great dangers. For we lose the overview of the whole, required for any mental activity aiming at discovering something essentially new or even just essentially new combinations of old ideas. In order to meet this drawback as far as possible it may be useful if from time to time a single individual who is occupied with the work of scientific detail should try to give a larger and scientifically educated public a survey of the branch of knowledge in which he is working." 1)

This book is addressed to those who share these thoughts of that great physicist and thinker, Ludwig Boltzmann.

NOTE

1. L. Boltzmann, Populäre Schriften. Verlag Johann Ambrosius Barth, 1905, pages 198, 199. This translation is from the authorized translation: L. Boltzmann, Theoretical physics and philosophical problems, D. Reidel Publ. Co., 1974, p. 77.

TABLE OF CONTENTS

INTRODUCTION

> "... toutes les sciences réunies ne sont rien autre chose que l'intelligence humaine, qui reste toujours une, toujours la même, si variés que soient les sujets auxquels elle s'applique et qui n'en reçoit pas plus de changements que n'en apporte à la lumière du soleil la variété des objets qu'elle éclaire ..." Descartes

There exist a good many, sometimes excellent, books which cover the theory and methods of solving mathematical programming problems and problems from mathematical economics. This book is characterized by the fact that it contains an exposition of some of the results of these sciences as a consequence of the fundamental principles of analytical mechanics and thermodynamics. It is addressed to a wide and diverse audience, which explains the author's decision to use only a relatively narrow collection of results known to everyone interested in mechanics or physics or familiar from the standard courses taught in institutions of higher learning.

The author shares the thoughts of Lanczos on mechanics [40]:

> "There is hardly any other branch of the mathematical sciences in which abstract mathematical speculation and concrete physical evidence go so beautifully together and complement each other so perfectly. It is no accident that the principles of mechanics had the greatest fascination for many of the outstanding figures of mathematics and physics. ... Analytical mechanics is much more than an efficient tool for the solution of dynamical problems that we encounter in physics and engineering." 1)

Lagrange, Laplace, Euler, and Hamilton have erected an edifice whose foundations were laid by Galileo, Leibnitz, and Newton, which is, according to Boltzmann, "a magnificent example for every physics-mathematical theory". Thus, analytical mechanics has also been a model for the creators of classical thermodynamics and one of its most important chapters: thermodynamic potential theory.

The fact that the general principles of analytical mechanics and thermodynamics play a role in all physical and chemical disciplines naturally invites us to use the same methods to study the behaviour of all material systems, with economic systems included.

The author does not pretend that this approach will lead us to a new economic theory. This book rather exemplifies an inclination which can hardly be defined better than was done by Von Neumann and Morgenstern:

> "... It is without doubt reasonable to discover what has led to progress in other sciences, and to investigate whether the application of the same principles may not lead to progress in economics also. Should the need for the application of different principles arise, it could be revealed only in the course of the actual development of economic theory. This would in itself constitute a major revolution. But since most assuredly we have not yet reached such a state - and it is by no means certain that there ever will be need for entirely different scientific principles - it would be very unwise to consider anything else than the pursuit of our problems in the manner which has resulted in the establishment of physical science." [50]. [2)]

This book describes the physical interpretations of mathematical programming problems and of certain economical models. Because the equilibrium state of the equivalent physical models defines the optimal bivector of the two (dual) associated problems, the equilibrium conditions of mechanics naturally express the fundamental duality theorem.

The models we study are subjected to rigid or elastic constraints and contain, as a moving force, an incompressible liquid or a perfect gas. One develops a general method for numerical solutions by means of the fundamental idea of controlling the transition of the physical system to an equilibrium state by the introduction of redundant and artificial constraints which are independent of time.

The reader will see physical models of various problems which one meets in linear algebra, in linear and nonlinear programming, in equilibrium theory, and in economic growth theory. He will see that maximalization and the search for equilibrium prices are, in fact, equilibrium problems for active physical systems (in the first case) or passive, i.e., isolated, systems (in the second case).

Because the search for an equilibrium of an economy with several subsystems each with their own objectives is equivalent to the problem of equilibrium for a passive physical system, the idea follows that such an economy will have a global state function which will have its maximum in the equilibrium state. In analogy to the entropy of a physical system that function does not only define the equilibrium states but also the direction in which the economy evolves spontaneously. Thus, the possibility arises in economic research of using the fundamental ideas, concepts, principles, and methods of analytical mechanics and phenomenological and statistical thermodynamics.

We would like to devote a few words to the contents of the book to help the reader decide whether he should continue to read it. Let us remark in the first place that the physical and mechanical models considered naturally lead us to a treatment of problems of mathematical programming and mathematical economics in terms of the equilibria of physical systems of which the state function is an analogue of Leibnitz's forces function or Gibbs' thermodynamic potential. As to the constraints, their analogues are the bilateral or unilateral constraints which determine the admissible parameter variations in the state of the system under consideration. In this setting, it is clear that the basic conditions for optimality and economic equilibrium must be established by means of the principles of virtual displacement, which play such a fundamental role in various sciences - from analytical mechanics, calculus of variations and thermodynamics, to the theory of relativity.

Precisely this principle, essentially, lies behind all constructions of conditions for optimality in problems of mathematical programming, to which, just as in mechanical problems there apply the words of Lagrange:

> "... and, in general, I believe it possible to predict that all the general principles which can, maybe, still be discovered in the science of equilibrium will be nothing else but the same principle of virtual velocities (displacments) viewed in different ways and only differing in the ways in which it is expressed. But

this principle is not only extremely simple in itself and very general; it also has, in addition, the unique and precious advantage that it encompasses all problems which can be posed concerning the equilibrium of solid bodies" [38].

In the literature on mathematical programming one usually finds described the method of indeterminate multipliers (Lagrange multipliers). One should not forget that Lagrange proposed this method uniquely as a rule for applying the principle of virtual displacements (velocities) to problems of mechanics for systems subject to idealized constraints.

It was also Lagrange who interpreted these multipliers, which still bear his name, in physical terms and who formulated the principle of 'removing the constraints' (principe de la libération) which 'has the same generality as the principle of virtual velocities' (Bertrand) [38].

Recall what has been said concerning primal and dual problems, to wit that both constitute the same equilibrium problem where only the state parameters vary independently. This situation can be considered analogous to the statics of elastic systems when the state parameters are either constraints or constituant deformations of the system; that is, extensive or intensive state variables. Such a treatment of duality not only enriches it but it also makes it more accessible. It is a curious fact that the rules which describe the physical properties of bodies subject to constraints such as Hooke's law or the law of Clapeyron-Mendeleev also connect the optimal solutions of the primal and dual problems and that the equality of the values of the 'cost' functions for the two problems is an analogue of the law of conservation of energy. 3)

Thus the unknowns of the primal and dual problems are the parameters of the state of the system of which the equilibrium defines the optimal values of the two sets of parameters. For economic equilibrium problems this means that the prices are also parameters of the state of the economy, equally relevant as the quantitities of available goods and that a change in the external conditions will lead to a new equilibrium state in which prices and quantities of goods are connected by relations valid for all exterior conditions.

Economic systems have, as was mentioned before, a state function, analogous to entropy. The directions of growth of this function determine the possible directions of spontaneous evolution of the system and the maximum determines the equilibrium state. Consequently there arises the idea that for these economic systems there exist analogues of the thermodynamic inequalities from which there follows the principle of stability of Le Chatelier [36].

The book pays particular attention to new numerical techniques which result from the principles and methods of the general theory of equilibrium and which use the known physical properties of bodies subject to constraints which represent the optimization problems of mathematical programming, systems of equations and inequalities, and mathematical economics models.

In this connection the method of redundant constraints is central. At the basis of this method in turn lies the idea of controlling the evolution process of a physical system to an equilibrium state. This control is realized by means of a series of supplementary (redundant) constraints which are imposed and which were not present in the problem being modelled. By imposing these time-independent redundant constraints, adapted to the actual state of the system, one decomposes the big complicated process in a series of elementary transition processes between intermediate states corresponding to equilibria for the system with the additional redundant constraints. Because no work is done by the external forces when the (redundant) constraints are changed from one set to another the processes evolve spontaneously and the convergence of the 'redundant constraints algorithms' follow from the second law of thermodynamics.

We also discuss the method of a sequence of unconstrained minimizations which is also called the method of penalty functions or weight functions. When he discussed the difficulties adhering to the Rayleigh-Ritz method when applied to oscillation problems for elastic systems, Richard Courant suggested an idea for a method which may be used to overcome the difficulties which come from the presence of rigid boundaries. This idea of Courant can be formulated as follows [23]:

> "Quite generally rigid boundary conditions should be regarded as a limiting case of natural conditions in which a parameter tends to infinity. This corresponds to the physical fact that rigid constraints are only an idealized limiting case of very large restoring forces." 4)

This idea can be put to work in various ways. Courant