Modern Switching Theory and Digital Design

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Preface

Switching theory is recognized as the foundation for computer science and digital design. It is, therefore, no surprise that almost every major university in the nation offers a course, or a sequence of two courses, in the departments of computer science, electrical engineering, and/or applied mathematics. This course is usually a requirement for all graduate students who are pursuing an advanced degree in computer science, digital systems, or combinatory mathematics. This book is a compilation of classnotes of a course on switching theory taught by the author for the past eight years. Senior and graduate students taking this course were familiar with the following subjects:

1. Truth tables.
2. Logic gates.
3. Karnaugh maps.
4. Switching function minimization methods.
5. Flip-flops.
6. Basic digital devices, such as, counters, registers, basic binary adders and subtractors, etc.

This was, in fact, the only background required for this course. In selecting a textbook for this course, the author was a little surprised to find no textbooks on switching theory at this level which were published since 1970. However, there has been a tremendous amount of research in the area since 1970. It seemed apparent to this author that a textbook including these recent research results was needed. This book was born out of this need.

This book contains twelve chapters which comprise four parts:

1. Boolean Algebra and Boolean Differential Calculus (Chapters 1, 2, and 3).
2. Combinational Logic (Chapters 4, 5, and 6).
3. Sequential Logic (Chapters 7, 8, 9, and 10).
4. Digital Design (Chapters 11, 12).

A brief description by chapter is as follows.

Chapter 1 describes Boolean algebra and its properties. It provides the mathematical foundation of switching theory and digital design. A generalization of two-valued Boolean algebra or switching algebra is introduced in Chapter 2. In this generalized algebra, every element is presented by a binary vector and two new operations; the rotation operation and the generalized complement operation are defined. It is shown that DeMorgan's theorem, Shannon's theorem, and the expansion theorem are generalized into more general forms which include their corresponding ordinary versions as special cases. Chapter 3 introduces the partial derivative, the partial differential, the total differential, and the total variation of a Boolean function. Many properties of these differential and variational operators of Boolean functions are presented. Based on the partial derivative of a Boolean function, the MacLaurin expansion and the Taylor expansion of a Boolean function are derived; the expansions are analogous to those of real functions of real variables. In addition, two convenient ways for computing Boolean derivatives and differentials of a switching function are included.

Special switching functions are useful in switching circuit design. Four special functions, monotonic, threshold, symmetric, and functionally complete functions are presented in Chapter 4. Many properties of these functions and algorithms for determining them are included. Chapter 5 presents multivalued switching functions, particularly their analysis and realization. As the number of components in an integrated circuit (IC) chip increases, the need for circuit testing becomes a necessary part of the manufacturing process. Several methods for deriving fault detection experiments for single and multiple logic faults in combinational circuits are presented in Chapter 6. Two computer-oriented algorithms, D-ALGorithm version II (DALG-II) and TEST-DETECT, are included which are applicable to the fault detection test generation for large combinational circuits.

In Chapter 7, three important problems, the state minimization, the state assignment, and the machine decomposition of sequential machines, are solved by the use of the substitution property (S. P.) partition of the states. A systematic representation of the Rabin-Scott machine, known as the regular expression, is presented in Chapter 8. This includes both the deterministic and the nondeterministic sequential machines. Two types of equivalence, indistinguishability equivalence and tape equivalence, are discussed. It is shown that for deterministic sequential machines, the two imply each other. A systematic procedure for obtaining a regular expression from a transition diagram and a procedure for constructing a transition diagram from a regular expression are presented. In Chapter 9, it is shown that any sequential machine can be physically realized by a clocked sequential circuit and obtained by it. The realizations of sequential machines using the pulse-mode and fundamental-mode circuits are also studied in detail. The analysis and design of pulse-mode sequential circuits are similar to those of clocked sequential circuits. But the analysis and design of fundamental-mode sequential circuits are different from those of clocked sequential circuits. The two undesirable transient phenomena of fundamental circuits, race and hazards and their elimination, are discussed. Chapter 10 introduces a method for designing a fault-detection experiment for a sequential machine. It is shown that the problem of designing a fault-detection experiment is actually a restricted problem of machine identification. The construction of a fault-detection experiment consists of three phases: the initialization phase, the state identification phase, and the transition verification phase. They are described in detail and are illustrated by examples.

The last two chapters are devoted to the discussion of modern digital (circuits and systems) design. Chapter 11 describes digital design using digital integrated circuits. In particular, digital design using various types of MSI and LSI integrated circuits is presented. Chapter 12 presents the digital design using microprocessors—the state-of-art. The basic design procedure is outlined and the hardwares and softwares of commonly used microprocessors are presented. Several digital design microcomputer systems are used to illustrate this new digital design method.

Many of the materials included in this book are recent research results which have never been included in any textbooks.

It is the author's experience that students always welcome good examples, particularly in illustrating difficult concepts and theory. A special feature of this book is that it includes many such examples throughout. In order to make sure that the student not only understands the theory but also knows how to apply it, a large number of exercises are given at the end of almost every section.

A picture is worth a thousand words. Figures, tables, and flow-charts are given throughout the book to help the reader to "see through" the theory.

I would like to thank Dr. M. E. Van Valkenburg, Dr. K. S. Fu, Dr. H. S. Hayre, and Dr. M. S. Ghausi for their advice and friendship. Special thanks are due to Dr. W. R. Upthegrove and Dr. C. R. Haden for their encouragement and support. I am also thankful to Mrs. Mary-Allen Kanak for preparing the drawings of Chapters 11 and 12, and to Mr. Mike Weible for proofreading the manuscript.

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Contents

1

Boolean Algebra and Boolean Function

Switching theory deals primarily with the analysis (characterization, minimization, etc.) and synthesis (realization) of a special type of function, defined on a special type of algebra known as *switching algebra*. Switching algebra is, in turn, a special type of *Boolean algebra;* and the special type of function, known as the *switching function*, is a mapping defined on switching algebra. Switching algebra that contains two elements, 0 and 1, is the two-element Boolean algebra (the simplest nondegenerate Boolean algebra). To understand how switching algebra is derived, one must first learn Boolean algebra, its mathematical foundation. In fact, Boolean algebra is the mathematical foundation of the entire field of switching theory.

The algebraic structure of Boolean algebra is derived from the ordered set. We begin the chapter by introducing ordered sets and the one-to-one relationship between elements in set theory and elements in algebra. Before introducing Boolean algebra, we first define *lattice*, which is a special subclass of the class of ordered sets. Boolean algebra is a special class of a subclass of lattices known as the *complemented distributive lattice*, or the *Boolean lattice*. Important properties of Boolean algebra are discussed in detail. Finally, the formal definition of Boolean function and its canonical forms are presented. The existence of the canonical forms for every Boolean function provides us with a convenient means of determining the equivalence between two Boolean functions and with a basis for deriving switching-function minimization methods, which will be discussed in Chapter 2.

1.1 Sets, Ordered Sets, and Algebras

Set theory is often referred to as the "root" of mathematics. We can consider every branch of mathematics to be a study of sets of objects of one kind or another. For instance, roughly speaking, geometry is a study of sets of points. Algebra is concerned with sets of numbers and operations on those sets. Analysis deals mainly with sets of functions. The study of sets and their use in the foundations of mathematics was begun in the latter part of the nineteenth century by the German mathematician Georg

Cantor (1845–1918). Since then, set theory has unified many seemingly disconnected ideas and has, in an elegant and systematic way, helped to reduce many mathematical concepts to their logical foundations.

The objectives of this section are threefold. The first is to serve as a review of some of the relevant materials in set theory. The second is to study three types of ordering: partial ordering, total ordering (a special case of partial ordering), and well-ordering (a special case of total ordering), and their corresponding types of sets. The third is to show the analogous quantities among sets, ordered sets, and algebras.

A *set* is a collection of objects in which nothing special is assumed about the nature of the individual objects. The individual objects in the collection are called *elements* or *members* of the set, and they are said *to belong to* (or *to be contained in*) the set. A group of people, a bunch of flowers, and a sequence of numbers are examples of sets. Here, people, flowers, and numbers are elements or members of these sets. It is important to know that a set itself may also be an element of some other set. For example, a line is a set of points; the set of all lines in the plane is a set of sets of points. In fact, a set can be a set of sets of sets, and so on. Let A be a set and x and y be elements of A. Define the relation "$x \leq y$" as "y includes x" and the relation "$x < y$" as "y strictly includes x."

DEFINITION 1.1.1

A relation $\leq$ on a set A is said to be a *partial ordering* on A if it satisfies the following axioms.
(01) Reflexive: For all $x \in A$, $x \leq x$.
(02) Antisymmetric: If $x,y \in A$, $x \leq y$, and $y \leq x$, then $x = y$.
(03) Transitive: If $x,y,z \in A$, $x \leq y$, and $y \leq z$, then $x \leq z$.
A set P over which a relation $\leq$ of partial ordering is defined is called a *partially ordered set*, or a *poset*.

In the definition, the reason for qualifying "partial" is that some questions about order may be left unanswered.

DEFINITION 1.1.2

A relation R on A is said to be *connected* whenever $x,y \in A$ implies that $x \leq y$ or $y \leq x$.

From Definitions 1.1.1 and 1.1.2, we define:

DEFINITION 1.1.3

A relation $\leq$ on a set A is said to be a *total ordering* of A if (a) it is a partial ordering of A, and (b) in addition, it satisfies the following axiom:
(04) Connected: Whenever $x,y \in A$ implies that $x \leq y$ or $y \leq x$.
A set C over which a relation of total ordering is defined is called a *totally ordered set*, or a *simply ordered set*, or a *chain*.

As mentioned before, the elements of a set may themselves be sets. A special class of such sets is the power set.

DEFINITION 1.1.4

Let A be a given set. The *power set* of A, denoted by $P(A)$, is a family (set) of sets such that when $X \subseteq A$, then $X \in P(A)$. Symbolically, $P(A) = \{X \mid X \subseteq A\}$.

Example 1.1.1

The power set of the empty set $\varnothing$ is a singleton $\{\varnothing\}$.

Example 1.1.2

Let $A = \{a, b, c\}$. The power set of A is

$$P(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$$

THEOREM 1.1.1

Prove that if a set A has exactly n elements, then $P(A)$ will have exactly 2^n elements.

Proof: One way of proving this theorem is to tabulate the number of possible subsets of A as follows:

TABLE 1.1.1 Number of Possible Subsets of a Set A with n Elements

Number of elements contained in a subset of A	*Number of subsets*
0	C_0^n
1	C_1^n
.	.
.	.
.	.
n	C_n^n
$\left(C_i^n = \dfrac{n!}{i!(n-i)!}\right)$	

Thus, the total number of subsets of A is $C_0^n + C_1^n + \ldots + C_n^n$. From the binomial theorem,

$$(1 + x)^n = C_0^n + C_1^n x + C_2^n x^2 + \ldots + C_n^n x^n$$

where x is a real number and n is a positive integer. When we let $x = 1$ in the expression above we find that $C_0^n + C_1^n + C_2^n + \ldots + C_n^n = 2^n$. Hence, the theorem is proved.

Another, more intuitive proof may be given as follows: Each element of A is either in or is not in some subset. Thus, there are n independent binary choices, or 2^n ways to choose a subset. ∎†

†∎ shows the end of a proof.

We sometimes use the symbol $(A, \leq)$ to denote a poset (chain), where A is a set and $\leq$ is a partial (total) ordering relation in A. Before we proceed further, let us see some simple examples of posets and chains.

Example 1.1.3

Let A be a set. The set-theoretic inclusion relation $\subseteq$ is a partial ordering in the power set $P(A)$. It is a total ordering if A is an empty set or a singleton.

Example 1.1.4

Another interesting example of a relation of partial ordering is arithmetic divisibility. Let A be a set of divisors of 100: $A = \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$. Define a relation "$x \leq y$" as "$x$ is a divisor of y." We can indicate this relation among the elements of A by using an *inclusion diagram* (Fig. 1.1.1). It is obvious that the relation "x is a divisor of y" is a partial ordering relation in A, not a total ordering relation, because

2 and 5
4, 10, and 25
20 and 50

are not related by this relation.

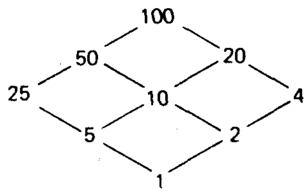


Fig. 1.1.1 The inclusion diagram of $(A, \leq)$ of Example 1.1.4.

Example 1.1.5

Now, if we consider the same relation "$x \leq y$" for "x is a divisor of y," but the set A is the set of divisors of 8, that is, $A = \{1, 2, 4, 8\}$, then the relation $\leq$ is a total-ordering relation in A, as shown in Fig. 1.1.2.

Fig. 1.1.2 The inclusion diagram of $(A, \leq)$ of Example 1.1.5.

In a poset A, if B is a nonempty subset of A and $b_0 \in B$, we call b_0 a *least element* of B if $b_0 \leq b$, for all $b \in B$; and we call b_0 a *minimal element* of B if there is no $b \in B$ such that $b < b_0$. A least element is necessarily minimal; but a minimal element need not be a least element, because a partial-ordering relation does not imply connected.

A *greatest element* and *maximal element* are defined in the corresponding way. By axiom 02, B can have at most one least and one greatest element, whereas it can have many minimal and maximal elements. The least and greatest elements of a poset are denoted by 0 and 1, respectively, whenever they exist.

From the definition of partial ordering, the following consequence is immediate.

THEOREM 1.1.2

Any finite subset X of a poset P has minimal and maximal elements.

Proof: Suppose that X is a singleton $X = \{x_1\}$. By the first condition of partial ordering, $x_1 \leq x_1$, x_1 may be considered as both a minimal element and a maximal element in X. If X contains two elements, $X = \{x_1, x_2\}$. There are two possible cases. One is that x_1, x_2 are related (i.e., either $x_1 \leq x_2$ or $x_2 \leq x_1$); hence, one is a minimal element and the other is a maximal element. Another case is that x_1, x_2 are unrelated. In this case, x_1, x_2 may be considered as both minimal and maximal elements in X. The induction of this argument for X containing n finite number of elements should be clear. ▮

From the definition of total ordering, it follows immediately that

THEOREM 1.1.3

For a chain, the notions of minimal and least (maximal and greatest) are equivalent. Hence, any finite chain has a least (first) and a greatest (last) element.

Proof: Since a least (greatest) element is necessarily minimal (maximal), we need only show that in a chain, a minimal (maximal) element is also a least (greatest) element of the chain. Let C be a chain. By Theorem 1.1.2, C has minimal and maximal elements. Let a be a minimal element of C (i.e., there is no x in C such that $x < a$). By axiom 04, $x \geq a$ for every x in A. Hence, a is also a least element of C. The proof that a maximal element of A is also a greatest element of A follows similarly. ▮

Example 1.1.6

In Example 1.1.4, consider a subset B of A, $B = \{4, 5, 10, 20, 25, 50\}$, as shown in Fig. 1.1.3. It is clear that B is a poset and has minimal elements 4 and 5 and maximal elements 20 and 50.

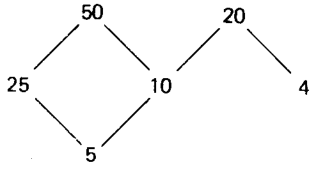


Fig. 1.1.3 The inclusion diagram of $(B, \leq)$ of Example 1.1.6.

Note that in Example 1.1.5 the element 1 is the least (first) element and 8 is the greatest (last) element.

DEFINITION 1.1.5

Let S and T be two sets (algebraic systems). If there exists a one-to-one correspondence between S and T, the correspondence is called an *isomorphism*, S and T are said to be isomorphic, or one set (system) is said to be *isomorphic* to the other. If the correspondence is not one-to-one, but many-to-one from S to T, the correspondence is called a *homomorphism*, or S is said to be *homomorphic* to T.

From Theorem 1.1.3 it follows that

THEOREM 1.1.4

Every finite chain of k elements is isomorphic to the ordered set $N_k = \{1, 2, \ldots, k\}$, where k is a positive integer. In other words, there always exists a mapping f from a chain C of k elements to N_k.

Proof: By Theorem 1.1.3 in C there exists a least and a greatest element. Let f map the least element of C to 1 and the least of the remaining elements into 2, and so on. Since both the chain C and N_k have the same number of elements, the greatest element of C, in this way, will be mapped to the greatest element k in N_k. Hence, the theorem is proved. ∎

The algebraic structure of a poset may be extended to a set of ordered pairs.

THEOREM 1.1.5

Let P be the Cartesian product of two posets A and B. Define an inclusion relation as

$$(a_1, b_1) \leq (a_2, b_2) \quad \text{iff } a_1 \leq a_2 \text{ in } A \text{ and } b_1 \leq b_2 \text{ in } B$$

The set P with the product-inclusion relation is a poset. More generally, if P is the Cartesian product $A_1 \times A_2 \times \ldots \times A_n$ with an ordering relation defined as $(a_1, a_2, \ldots, a_n) \leq (a_1', a_2', \ldots, a_n')$ iff $a_1 \leq a_1'$ in A_1, $a_2 \leq a_2'$ in $A_2, \ldots, a_n \leq a_n'$ in A_n, then the set P with the product-inclusion relation is a poset.

Proof: The proof is evident.

Example 1.1.7

Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 10, 20, 25, 50\}$ be two posets with ordering relation defined by arithmetic divisibility. The inclusion diagram for A is shown in Fig. 1.1.4; the inclusion diagram of B was shown in Fig. 1.1.3. The inclusion diagram of the Cartesian product $A \times B$ with the product-inclusion relation defined above is given in Fig. 1.1.5(a), and

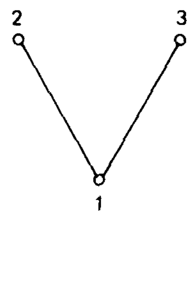
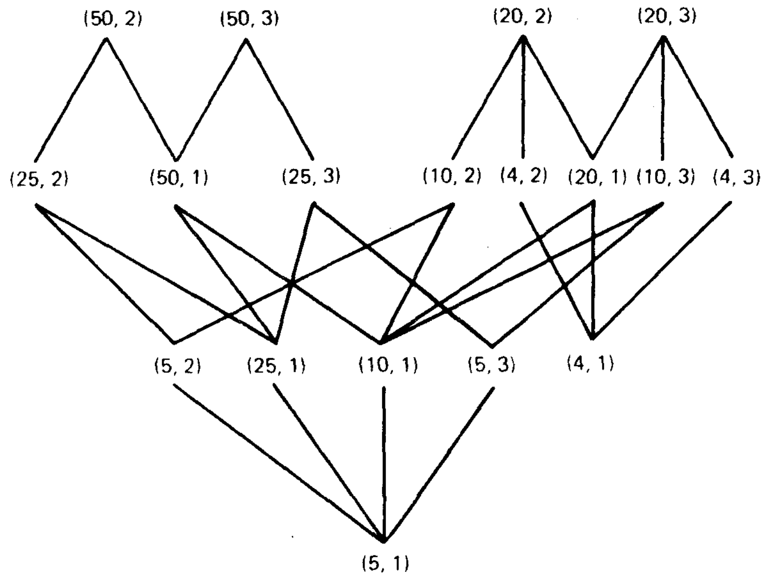
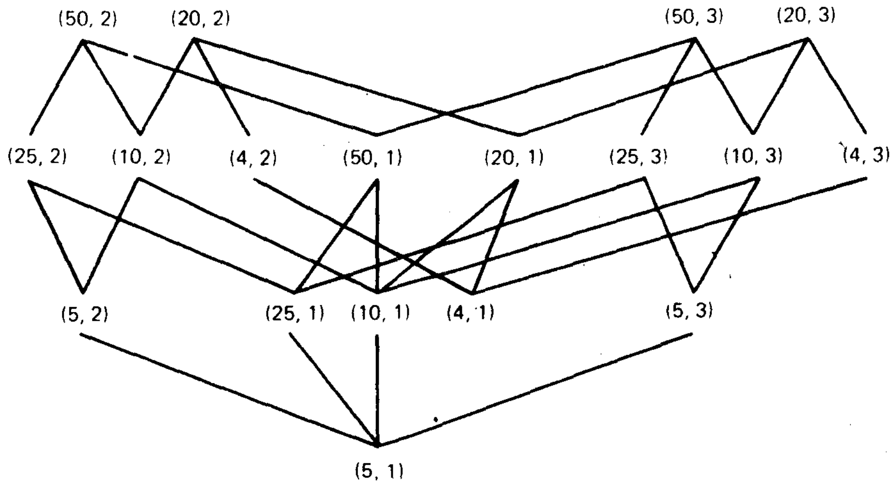


Fig. 1.1.4 The inclusion diagram of $(A, \leq)$ of Example 1.1.7.



(a)



(b)

Fig. 1.1.5 (a) The inclusion diagram of the Cartesian product A × B of Example 1.1.7. (b) A more systematic way of generating the inclusion diagram of the Cartesian product $A \times B$ of Example 1.1.7.

a more systematic way of generating this inclusion diagram is shown in Fig. 1.1.5(b). From these diagrams, it is seen that the set $A \times B$ is a poset.

Now we introduce the third type of ordering.

DEFINITION 1.1.6

A relation $\leq$ on a set A is said to be a *well-ordering* of A if (a) it is a total ordering of A, and (b) it is such that every nonempty subset of A has a least element.

Here are several simple examples of well-ordered sets. In the first example, we demonstrate that a totally ordered set need not be well ordered.

Example 1.1.8

Let Q, Z, Z_e, and Z_o be the sets of rational numbers, integers, even integers, and odd integers, respectively. The set R^1 of all real numbers is totally ordered by the arithmetical relation $\leq$; but neither in R^1 nor in its subsets Q, Z, Z_e, Z_o, etc., is there any least element.

Example 1.1.9

The set N of natural numbers with the arithmetical relation $\leq$ is a well-ordered set.

Example 1.1.10

The set $P = \{\{A_1\}, \{A_1, A_2\}, \{A_1, A_2, A_3\}, \ldots\}$ with the set-theoretic inclusion relation $\subseteq$ is a well-ordered set.

Now we want to show that ordered sets provide a link between sets and algebras. We begin this discussion with the following definition.

DEFINITION 1.1.7

Let P be a poset and x and y be two elements of A. An element b in P is said to be a *lower bound* of x and y if $b \leq x$ and $b \leq y$. An element m in P is said to be a *greatest lower bound* (g.l.b.) of x and y if $m \leq x$ and $m \leq y$, $b \leq m$ for all b such that $b \leq x$ and $b \leq y$. Dually, we define an element u in P to be an upper bound of x and y if $x \leq u$ and $y \leq u$, and an element l in P to be a *least upper bound* (l.u.b.) of x and y if $x \leq l$ and $y \leq l$, $l \leq u$ for all u such that $x \leq u$ and $y \leq u$.

Now we define:

DEFINITION 1.1.8

An element m in P is a *meet* of x and y if it is a g.l.b. of x and y. An element l in P is a *join* of x and y if it is a l.u.b. of x and y. We shall denote the meet and join of x and y by $m = x \cap y$ and $l = x \cup y$, respectively.†

†The symbols "$\cap$" and "$\cup$" are called "cap" and "cup" by some authors.

It is worth noting that

1. For given x and y, the meet and join are *unique* if they exist.
2. Meet and join are *order duals* of each other. By order dual, we mean that the definition of meet may be obtained by the definition of join simply by replacing the relation $x \leq y$ by its converse, and vice versa.

The second property is obvious from Definitions 1.1.7 and 1.1.8. The proof of the first property is as follows. Suppose that m and m' are both meets of x and y. Definition 1.1.7 implies that $m \leq m'$ and $m' \leq m$. By the antisymmetry axiom of partial ordering, m' should be equal to m. Hence, a meet, when it exists, is unique. By a similar argument, a join of x and y is also unique when it exists. We shall use the symbols O and I to denote the (unique) least and greatest elements of a partially ordered set whenever they exist.

We can show, among other properties, that the meet and join operations satisfy the absorption property:

$$x \cap (x \cup y) = x$$
$$x \cup (x \cap y) = x$$

The proofs of these identities will be given in the next section. Here, we just use them as examples to show the analogy between sets and algebras. The role that ordered sets play in linking sets and algebras is shown in Table 1.1.2.

TABLE 1.1.2 From Sets to Algebras

		Algebras	
Sets	*Ordered sets*	*Numbers or symbols in general*	*Integers or rational numbers*
A	x	a	a
B	y	b	b
$\cap$	$\cap$	$\cdot$	$\cdot$
$\cup$	$\cup$	$+$	$+$
$A \cap (A \cup B) = A$	$x \cap (x \cup y) = x$	$a \cdot (a + b) = \text{g.l.b.}[a, \text{l.u.b.}(a, b)] = a$	$a \cdot (a + b) = \min [a, \max (a, b)] = a$
$A \cup (A \cap B) = A$	$x \cup (x \cap y) = x$	$a + a \cdot b = \text{l.u.b.}[a, \text{g.l.b.}(a, b)] = a$	$a + a \cdot b = \max [a, \min (a, b)] = a$

The analogous quantities among sets, ordered sets, and algebras are tabulated in Table 1.1.3.