Problems in

Differential Geometry and Topology

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TO THE READER

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На английском языке

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СБОРНИК ЗАДАЧ ПО ДИФФЕРЕНЦИАЛЬНОЙ ГЕОМЕТРИИ И ТОПОЛОГИИ

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Contents

Preface

This book of problems is the result of a course in differential geometry and topology, given at the mechanics-and-mathematics department of Moscow State University. It contains problems practically for all sections of the seminar course. Although certain textbooks and books of problems indicated in the bibliography list were used in preparation of this volume, a considerable number of the problems were prepared for this book expressly.

The material is distributed over the sections as in textbook [3]. Some problems, however, touch upon topics outside the lectures. In these cases, the corresponding sections are supplied with additional definitions and explanations.

In conclusion, the authors express their sincere gratitude to all those who helped to publish this work.

ИЗДАТЕЛЬСТВО МИР

1
Application of Linear Algebra to Geometry

1.1. Prove that a vector set $\mathbf{a}_1, \ldots, \mathbf{a}_k$ in a Euclidean space is linearly independent if and only if

$$\det \|(\alpha_i, \alpha_j)\| \neq 0.$$

1.2. Find the relation between a complex matrix A and the real matrix rA of the complex linear mapping.

1.3. Find the relations between

$$\det A \text{ and } \det rA, \operatorname{Tr} A \text{ and } \operatorname{Tr} rA, \det (A - \lambda E) \text{ and } \det (rA - \lambda A).$$

1.4. Find the relation between the invariants of the matrices A, B and $A \oplus B$, $A \otimes B$.

Consider the cases of det and Tr.

1.5. Prove the formula

$$\det e^A = e^{\operatorname{Tr} A}.$$

1.6. Prove that

$$e^A e^B = e^{(A+B)} + C'[A, B]C''$$

for a convenient choice of the matrices C' and C'', where $[A, B] = AB - BA$.

1.7. Prove that if A is a skewsymmetric matrix, then e^A is an orthogonal matrix.

1.8. Prove that if A is a skewhermitian matrix, then e^A is a unitary matrix.

1.9. Prove that if $[A, A^*] = 0$, then the matrix A is similar to a diagonal one.

1.10. Prove that a unitary matrix is similar to a diagonal one with eigenvalues whose moduli equal unity.

1.11. Prove that a hermitian matrix is similar to a diagonal one with real eigenvalues.

1.12. Prove that a skewhermitian matrix is similar to a diagonal one with imaginary eigenvalues.

1.13. Let $A = \|a_{ij}\|$ be a matrix of a quadratic form, and $D_k = \det \|a_{ij}\|_{1 \le i,j \le k}$.

Prove that A is positive definite if and only if for all k, $1 \leqslant k \leqslant n$, the inequalities $D_k > 0$ are valid.

1.14. With the notation of the previous problem, prove that a matrix A is negative definite if and only if for all k, $1 \leqslant k \leqslant n$, the inequality $(-1)^k D_k > 0$ holds.

1.15. Put $\|A\|^2 = \sum_{i,k} |a_{ik}|^2$. Prove the inequalities

$$\|A + B\| \leqslant \|A\| + \|B\|,$$

$$\|\lambda A\| \leqslant |\lambda| \cdot \|A\|,$$

$$\|AB\| \leqslant \|A\| \cdot \|B\|.$$

1.16. Prove that if $A^2 = E_n$, then the matrix A is similar to the matrix $\begin{pmatrix} E_k & 0 \\ 0 & -E_l \end{pmatrix}$, $k + l = n$.

1.17. Prove that if $A^2 = -E$, then the order of the matrix A is $(2n \times 2n)$, and it is similar to a matrix of the form $\begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$.

1.18. Prove that if $A^2 = A$, then the matrix A is similar to a matrix of the form $\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$.

1.19. Prove that varying continuously a quadratic form from the class of non-singular quadratic forms does not alter the signature of the form.

1.20. Prove that varying continuously a quadratic form from the class of quadratic forms with constant rank does not alter its signature.

1.21. Prove that any motion of the Euclidean plane $\mathbf{R}^2$ can be resolved into a composition of a translation, reflection in a straight line, and rotation about a point.

1.22. Prove that any motion of the Euclidean space $\mathbf{R}^3$ can be resolved into a composition of a translation, reflection in a plane and rotation about a straight line.

1.23. Generalize Problems 1.21 and 1.22 for the case of the Euclidean space $\mathbf{R}^n$.

2
Systems of Coordinates

A set of numbers $q^1, q^2, \ldots, q^n$ determining the position of a point in the space $\mathbf{R}^n$ is called its curvilinear coordinates. The relation between the Cartesian coordinates $x_1, x_2, \ldots, x_n$ of this point and curvilinear coordinates is expressed by the equalities

$$x_s = x_s(q^1, q^2, \ldots, q^n), \tag{1}$$

or, in vector form, by

$$\mathbf{r} = \mathbf{r}(q^1, q^2, \ldots, q^n),$$

where $\mathbf{r}$ is a radius vector. Functions (1) are assumed to be continuous in their domain and to have continuous partial derivatives up to the third order inclusive. They must be uniquely solvable with respect to q^1, $q^2, \ldots, q^n$; this condition is equivalent to the requirement that the Jacobian

$$J = \left| \frac{\partial x_s}{\partial q^k} \right| \tag{2}$$

should not be equal to zero. The numeration of the coordinates is assumed to be chosen so that the Jacobian is positive.

Transformation (1) determines n families of the coordinate hypersurfaces $q^r = q^r_0$. The coordinate hypersurfaces of one and the same family do not intersect each other if condition (2) is fulfilled.

Owing to condition (2), any $n - 1$ coordinate hyperplanes which belong to different families meet in a certain curve. They are called coordinate curves or coordinate lines.

The vectors $\mathbf{r}_k = \dfrac{\partial \mathbf{r}}{\partial q^k}$ are directed as the tangents to the coordinate lines. They determine the infinitesimal vector

$$d\mathbf{r} = \sum_{k=1}^{n} \mathbf{r}_k dq^k$$

in a neighbourhood of the point $M(q^1, q^2, \ldots, q^n)$. The square of its length, if expressed in terms of curvilinear coordinates, can be found from the equality

$$ds^2 = (d\mathbf{r}, d\mathbf{r}) = \left(\sum_{s=1}^{n} \mathbf{r}_s dq^s, \sum_{k=1}^{n} \mathbf{r}_k dq^k \right) = \sum_{s,\ k=1}^{n} g_{sk} dq^s dq^k,$$

where (,) is the scalar product defined in $\mathbf{R}^n$.

The quantities $g_{sk} = g_{ks} = (\mathbf{r}_s, \mathbf{r}_k)$ define a metric in the adopted coordinate system.

An orthogonal curvilinear coordinate system is one for which

$$g_{sk} = (\mathbf{r}_s, \mathbf{r}_k) = \begin{cases} 0, & s \neq k \\ H_s^2, & s = k \end{cases}.$$

The quantities H_s^2 are called the Lamé coefficients. Thev are equal to the moduli of the vectors $\mathbf{r}_s$:

$$H_s = |\mathbf{r}_s| = \sqrt{\left(\frac{\partial x_1}{\partial q^s}\right)^2 + \left(\frac{\partial x_2}{\partial q^s}\right)^2 + \ldots + \left(\frac{\partial x_n}{\partial q^s}\right)^2}.$$

The square of the linear element in orthogonal curvilinear coordinates is given by the expression

$$ds^2 = H_1^2 dq^{1^2} + H_2^2 dq^{2^2} + \ldots + H_n^2 dqn^2.$$

2.1. Calculate the Jacobian $J = \left|\frac{\partial x_s}{\partial q^k}\right|$ of transition from Cartesian coordinates $(x_1, \ldots, x_n)$ to orthogonal curvilinear coordinates $(q^1, q^2, \ldots, q^n)$ in the space $\mathbf{R}^n$.

2.2. Calculate the gradient grad f of the function $f: \mathbf{R}^3 \to \mathbf{R}$ in an orthogonal curvilinear coordinate system.

2.3. Calculate the divergence div $\mathbf{a}$ of a vector $\mathbf{a} \in \mathbf{R}^3$ in an orthogonal curvilinear coordinate system.

2.4. Find the expression for the Laplace operator Δf of the function $f: \mathbf{R}^3 \to \mathbf{R}$ in an orthogonal curvilinear coordinate system.

2.5. *Cylindrical coordinates in* $\mathbf{R}^3$

$$q^1 = r, \quad q^2 = \varphi, \quad q^3 = z$$

are related to Cartesian coordinates by the formulae

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z.$$

(a) Find the coordinate surfaces of cylindrical coordinates.
(b) Compute the Lamé coefficients.
(c) Find expression for the Laplace operator in cylindrical coordinates.

2.6. *Spherical coordinates in* $\mathbf{R}^3$

$$q^1 = r, \quad q^2 = \theta, \quad q^3 = \varphi$$

are related to rectangular coordinates by the formulae

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta.$$

(a) Find the coordinate surfaces of spherical coordinates.
(b) Compute the Lamé coefficients.
(c) Find expression for the Laplace operator in spherical coordinates.

2.7. *Elliptic coordinates in* $\mathbf{R}^3$

$$q^1 = \lambda, \quad q^2 = \mu, \quad q^3 = z$$

are defined by the formulae

$$x = c\lambda\mu, \quad y = c\sqrt{(\lambda^2 - 1)(1 - \mu^2)}, \quad z = z,$$

where c is a scale factor.

(a) Find the coordinate surfaces of elliptic coordinates.

(b) Compute the Lamé coefficients.

2.8. *Parabolic coordinates in* $\mathbf{R}^3$

$$q^1 = \lambda, \quad q^2 = \mu, \quad q^3 = z$$

are related to Cartesian by the formulae

$$x = \frac{1}{2}(\mu^2 - \lambda^2), \quad y = \lambda\mu, \quad z = z.$$

(a) Express parabolic coordinates in terms of cylindrical.

(b) Find the coordinate surfaces of parabolic coordinates.

(c) Compute the Lamé coefficients.

2.9. *Ellipsoidal coordinates in* $\mathbf{R}^3$ are introduced by the equations ($a > b > c$):

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1 \ (\lambda > -c^2) \text{ (ellipsoid)},$$

$$\frac{x^2}{a^2 + \mu} + \frac{y^2}{b^2 + \mu} + \frac{z^2}{c^2 + \mu} = 1 \ (-c^2 > \mu > -b^2) \text{ (hyperboloid of}$$

one sheet),

$$\frac{x^2}{a^2 + \nu} + \frac{y^2}{b^2 + \nu} + \frac{z^2}{c^2 + \nu} = 1 \ (-b^2 > \nu > -a^2) \text{ (hyperboloid of}$$

two sheets).

Only one set of values λ, μ, ν corresponds to each point $(x, y, z) \in \mathbf{R}^3$.

The parameters

$$q^1 = \lambda, \quad q^2 = \mu, \quad q^3 = \nu$$

are called ellipsoidal coordinates.

(a) Express Cartesian coordinates x, y, z in terms of ellipsoidal coordinates λ, μ, ν.

(b) Compute the Lamé coefficients.

(c) Find expression for the Laplace operator in terms of ellipsoidal coordinates.

2.10. *Degenerate ellipsoidal coordinates* (α, β, φ) *in* $\mathbf{R}^3$ *for a prolate ellipsoid of revolution are defined by the formulae*

$$x = c \sin\beta \cos\varphi, \quad y = c \sinh\alpha \sin\beta \sin\varphi, \quad z = c \cosh\alpha \cos\beta,$$

where c is a scale factor, $0 \leqslant \alpha < \infty$, $0 \leqslant \beta < \pi$, $-\pi < \varphi \leqslant \pi$.

(a) Find the coordinate surfaces in this coordinate system.

(b) Compute the Lamé coefficients.

(c) Find expression for the Laplace operator.

2.11. *Degenerate ellipsoidal coordinate system* (α, β, φ) *in* $\mathbf{R}^3$ *for an oblate ellipsoid of revolution* is defined by the formulae

$$x = c \cosh\alpha \sin\beta \cos\varphi, \quad y = c \cosh\alpha \sin\beta \sin\varphi,$$

$$z = c \cosh\alpha \cos\varphi,$$

$$0 \leqslant \alpha < \infty, \quad 0 \leqslant \beta \leqslant \pi, \quad -\pi < \varphi \leqslant \pi.$$

(a) Find the coordinate surfaces for this coordinate system.

(b) Compute the Lamé coefficients.

(c) Find expression for the Laplace operator.

2.12. *Toroidal coordinate system* (α, β, φ) *in* $\mathbf{R}^3$ *is defined by the formulae*

$$x = \frac{c \sinh\alpha \cos\varphi}{\cosh\alpha - \cos\beta}, \; y = \frac{c \sinh\alpha \sin\varphi}{\cosh\alpha - \cos\beta}, \; z = \frac{c \sin\beta}{\cosh\alpha - \cos\beta},$$

where c is a scale factor, $0 \leqslant \alpha < \infty$, $-\pi < \beta \leqslant \pi$, $-\pi < \varphi \leqslant \pi$.

(a) Find the coordinate surfaces in a toroidal coordinate system.

(b) Compute the Lamé coefficients.

(c) Find expression for the Laplace operator.

2.13. *Bipolar coordinates in* $\mathbf{R}^3$

$$q^1 = \alpha, \quad q^2 = \beta, \quad q^3 = z$$

are related to Cartesian coordinates x, y, z by the formulae

$$x = \frac{a \sinh\alpha}{\cosh\alpha - \cos\beta}, \quad y = \frac{a \sin\beta}{\cosh\alpha - \cos\beta}, \quad z = z,$$

where a is a scale factor.

Compute the Lamé coefficients for a bipolar coordinate system.

2.14. *Bispherical coordinates in* $\mathbf{R}^3$

$$q^1 = \alpha, \quad q^2 = \beta, \quad q^3 = \varphi$$

are defined by the formulae

$$x = \frac{c \sin\alpha \cos\varphi}{\cosh\beta - \cos\alpha}, \quad y = \frac{c \sin\alpha \sin\varphi}{\cosh\beta - \cos\alpha}, \quad z = \frac{c \sinh\beta}{\cosh\beta - \cos\alpha},$$

where c is a constant factor, $0 \leqslant \alpha < \beta$, $-\infty < \beta < \infty$, $-\pi < \varphi \leqslant \pi$.

These formulae can be written shorter:

$$z + i\varrho = ci \cot \frac{\alpha + i\beta}{2} \quad (\varrho = \sqrt{x^2 + y^2}).$$

(a) Find the coordinate surfaces in a bispherical coordinate system.
(b) Compute the Lamé coefficients.
(c) Find expression for the Laplace operator.

2.15. *Prolate spheroidal coordinates in* $\mathbf{R}^3$

$$q^1 = \lambda, \quad q^2 = \mu, \quad q^3 = \varphi$$

are defined by the formulae

$$x = c\lambda\mu, \quad y = c\sqrt{(\lambda^2 - 1)(1 - \mu^2)}\cos\varphi,$$

$$z = \sqrt{c(\lambda^2 - 1)(1 - \mu^2)}\sin\varphi,$$

where $\lambda \geqslant 1$, $-1 \leqslant \mu \leqslant 1$, $0 \leqslant \varphi \leqslant 2\pi$, and c is a constant factor.
Compute the Lamé coefficients for this coordinate system.

2.16. *Oblate spheroidal coordinates in* $\mathbf{R}^3$

$$q^1 = \lambda, \quad q^2 = \mu, \quad q^3 = \varphi$$

are defined by the formulae

$$x = c\lambda\mu \sin\varphi, \quad y = c\sqrt{(\lambda^2 - 1)(1 - \mu^2)}, \quad z = c\lambda\mu \cos\varphi,$$

$$\lambda \geqslant 1, \quad -1 \leqslant \mu \leqslant 1, \quad 0 \leqslant \varphi \leqslant 2\pi.$$

Compute the Lamé coefficients for an oblate spheroidal coordinate system.

2.17. *Paraboloidal coordinates in* $\mathbf{R}^3$

$$q^1 = \lambda, \quad q^2 = \mu; \quad q^3 = \varphi$$

are defined by the relations

$$x = \lambda\mu \cos\varphi, \quad y = \lambda\mu \sin\varphi, \quad z = \frac{1}{2}(\lambda^2 - \mu^2).$$

(a) Compute the Lamé coefficients for a paraboloidal coordinate system.
(b) Find the coordinate surfaces.

2.18. Let H_1, H_2, H_3 be the Lamé coefficients for a certain curvilinear coordinate system in $\mathbf{R}^3$.
Prove the relations

$$(1)\ \frac{\partial}{\partial q^1}\frac{1}{H_1}\frac{\partial H_2}{\partial q^1} + \frac{\partial}{\partial q^2}\frac{1}{H_2}\frac{\partial H_1}{\partial q^2} + \frac{1}{H_3^2}\frac{\partial H_1}{\partial q^3}\frac{\partial H_2}{\partial q^3} = 0;$$

(2) $\frac{\partial}{\partial q^2}\frac{1}{H_2}\frac{\partial H_3}{\partial q^2}+\frac{\partial}{\partial q^3}\frac{1}{H_3}\frac{\partial H_2}{\partial q^3}+\frac{1}{H_1^2}\frac{\partial H_2}{\partial q^1}\frac{\partial H_3}{\partial q^1}=0;$

(3) $\frac{\partial}{\partial q^3}\frac{1}{H_3}\frac{\partial H_1}{\partial q^3}+\frac{\partial}{\partial q^1}\frac{1}{H_1}\frac{\partial H_3}{\partial q^1}+\frac{1}{H_2^2}\frac{\partial H_3}{\partial q^2}\frac{\partial H_1}{\partial q^2}=0;$

(4) $\frac{\partial^2 H_1}{\partial q^2\partial q^3}=\frac{1}{H_3}\frac{\partial H_3}{\partial q^2}\frac{\partial H_1}{\partial q^3}+\frac{1}{H_2}\frac{\partial H_1}{\partial q^2}\frac{\partial H_2}{\partial q^3};$

(5) $\frac{\partial^2 H_2}{\partial q^3\partial q^1}=\frac{1}{H_1}\frac{\partial H_1}{\partial q^3}\frac{\partial H_2}{\partial q^1}+\frac{1}{H_3}\frac{\partial H_2}{\partial q^3}\frac{\partial H_3}{\partial q^1};$

(6) $\frac{\partial^2 H_3}{\partial q^1\partial q^2}=\frac{1}{H_2}\frac{\partial H_2}{\partial q^1}\frac{\partial H_3}{\partial q^2}+\frac{1}{H_1}\frac{\partial H_3}{\partial q^1}\frac{\partial H_1}{\partial q^2}.$

2.19. Prove that if functions $H_1(q^1, q^2, q^3)$, $H_2(q^1, q^2, q^3)$, $H_3(q^1, q^2, q^3)$ of class C^3 satisfy the relations of the previous problem, then they are the Lamé coefficients for a certain transformation

$x_s = x_s(q^1, q^2, q^3)$, $s = 1, 2, 3$.

3
Riemannian Metric

3.1. Prove that the metric $ds^2 = dx^2 + f(x)dy^2$, $0 < f(x) < \infty$ can be transformed to the form $ds^2 = g(u, v)(du^2 + dv^2)$ (isothermal coordinates).

3.2. Prove that local isothermal coordinates can be defined on any real analytic surface M^2. Find the conformal representation of the metric ds^2.

3.3. Mercator's projection is defined as follows: rectangular coordinates (x, y) are defined on a map so that a constant bearing line (where the compass needle remains undeflected) on the earth's surface is put into correspondence with a straight line on the map.

(a) Prove that to a point on the surface of the globe with spherical coordinates (θ, φ) on the map, there corresponds, in Mercator's projection, the point with coordinates $x = \varphi$, $y = \ln \cot\theta/2$.

(b) How can the metric on the terrestrial globe be written in terms of the coordinates (x, y)?

3.4. Prove that the metric ds^2 on the standard hyperboloid of two sheets which is embedded in the pseudo Euclidean space $\mathbf{R}^3_1$ coincides with the metric on the Lobachevski plane

3.5. Write the metric on the sphere S^2 in complex form.

3.6. Find a metric on the two-dimensional space of velocities in relativity theory.

3.7. Change the coordinates in the previous problem so that $v \to \tanh\chi$ (where v is the velocity of the moving point).

3.8. Write the metric of the previous problem in polar coordinates for the unit circle.

3.9. Calculate the length of a circumference and the area of a circle on (a) the Euclidean plane, (b) a sphere, (c) the Lobachevski plane.

3.10. Let the Lobachevski plane be realized as the upper half-plane of the Euclidean plane. We call Euclidean semicircumferences with centres on the axis Ox and Euclidean half-lines resting upon the axis Ox and orthogonal to it "straight lines" of the Lobachevski plane. We call a figure formed by three points and the segments of "the straight lines" joining them a triangle in the Lobachevski plane.

Prove that the sum of the angles of a triangle in the Lobachevski plane is less than π.

3.11. (Continuation of Problem 3.10.) Let ABC be an arbitrary triangle in the Lobachevski plane, a, b, c the non-Euclidean lengths of the sides BC, AC, AB, and α, β, γ the values of its angles at the vertices A, B, C. Prove the following relations:

(1) $$\cosh a = \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma};$$

(2) $$\cosh b = \frac{\cos\beta + \cos\gamma\cos\alpha}{\sin\gamma\sin\alpha};$$

(3) $$\cosh c = \frac{\cos\gamma + \cos\alpha\cos\beta}{\sin\alpha\sin\beta}.$$

3.12. (Continuation of Problem 3.11.) Prove the analogue of the law of sines for the Lobachevski plane:

$$\frac{\sinh a}{\sin\alpha} = \frac{\sinh b}{\sin\beta} = \frac{\sinh c}{\sin\gamma} = \frac{\sqrt{Q}}{\sin\alpha\sin\beta\sin\gamma},$$

where $Q = \cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma - 1$.

3.13. (Continuation of Problem 3.12.) Prove the following formulae expressing the angles of a triangle in the Lobachevski plane in terms of its sides:

(1) $$\cos\alpha = \frac{\cosh b\cosh c - \cosh a}{\sinh b\sinh c},$$

(2) $\cos\beta = \dfrac{\cosh c \cosh a - \cosh b}{\sinh c \sinh a}$,

(3) $\cos\gamma = \dfrac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b}$.

3.14. (Continuation of Problem 3.13.)
Assume that $\gamma = \pi/2$, i.e., the triangle ABC is right. Prove the following relations:

(1) $\sinh a = \sinh c \sin\alpha$;

(2) $\tanh a = \tanh c \cos\beta$;

(3) $\tanh a = \sinh b \tan\alpha$;

(4) $\cosh c = \cosh a \cosh b$;

(5) $\cosh c = \cot\alpha \cot\beta$;

(6) $\cosh a = \cos\alpha/\sin\beta$.

3.15. Let ABC be a spherical triangle on a sphere of radius R, α, β, γ the values of the angles at the vertices A, B, C and a, b, c the lengths of the sides BC, AC, AB. Prove the following relationship

$$\cos\frac{a}{R} = \cos\frac{b}{R}\cos\frac{c}{R} + \sin\frac{b}{R}\sin\frac{c}{R}\cos\alpha.$$

4
Theory of Curves

4.1. Let C be a plane curve, M_0 a point of the curve C, and XOY a rectangular system of coordinates given in the plane of the curve. Denote the points of intersection of the tangent and the normal to this curve with the axis OX by T and N, respectively. Let P be the projection of the point M_0 onto the axis OX.

(a) Find the equation of the curve C if its subnormal PN is constant and equal to a.

(b) Find the equation of the curve C if its subtangent PT is constant and equal to a.

(c) Find the equation of the curve C if the length of its normal M_0N is constant and equal to a (for any point M_0 on the curve).

4.2. Find the equation of the curve C whose tangent MT is constant in length and equal to a.