Mathematics and Its Applications

Vadim Komkov

Variational Principles of Continuum Mechanics with Engineering Applications

Volume 1: Critical Points Theory

D. Reidel Publishing Company

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EDITOR'S PREFACE

Approach your problems from the right end and begin with the answers. Then one day, perhaps you will find the final question.

'The Hermit Clad in Crane Feathers' in R. van Gulik's *The Chinese Maze Murders*.

It isn't that they can't see the solution. It is that they can't see the problem.

G.K. Chesterton. *The Scandal of Father Brown* 'The point of a Pin'.

Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the "tree" of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related.

Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as "experimental mathematics", "CFD", "completely integrable systems", "chaos, synergetics and large-scale order", which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics. This programme, Mathematics and Its Applications, is devoted to new emerging (sub)disciplines and to such (new) interrelations as exempla gratia:

- a central concept which plays an important role in several different mathematical and/or scientific specialized areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have and have had on the development of another.

The Mathematics and Its Applications programme tries to make available a careful selection of books which fit the philosophy outlined above. With such books, which are stimulating rather than definitive, intriguing rather than encyclopaedic, we hope to contribute something towards better communication among the practitioners in diversified fields.

Because of the wealth of scholarly research being undertaken in the Soviet Union, Eastern Europe, and Japan, it was decided to devote special attention to the work emanating from these particular regions. Thus it was decided to start three regional series under the umbrella of the main MIA programme.

A long time ago it was certainly time that mechanics (both particle mechanics and continuum mechanics) was a main source of mathematical problems and ideas. By the first half of this century, however, the limits between the two areas of research certainly had become rather tenuous. Also, a standard gap of some 50 years between the developments of theory and applications had developed.

Now things have changed again. A period of gathering in the harvest of some 70 years of intense and often intensely specialised developments has started. Also, the fifty-year gap is fast disappearing and has disappeared completely in some areas. Continuum mechanics is one of the areas profiting from all this. There is also a new awareness that the matter of "what can one really do with all these beautiful theoretical results and ideas" is an important part of research. This book testifies to the fact that abstract results and techniques are useful and can be put to effective use and are, given a good expositor, accessible to, in this case, engineers.

The unreasonable effectiveness of mathematics in science ...

Eugene Wigner

Well, if you know of a better 'ole, go to it.

Bruce Bairnsfather

What is now proved was once only imagined.

William Blake

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited.

But when these sciences joined company they drew from each other fresh vitality and thenceforward marched on at a rapid pace towards perfection.

Joseph Louis Lagrange.

Bussum, July 1985

Michiel Hazewinkel

INTRODUCTION

VARIATIONAL PRINCIPLES OF CONTINUUM MECHANICS

In writing this monograph, I had to consider the basic interplay between mathematics and mechanics. In particular one has to answer some obvious questions in considering the development of a mathematical theory which is primarily oriented towards an applied science. A majority of engineers or physicists would have given an obvious answer concerning the role of mathematics. It is used for solving problems. Modern physicists are not quite so certain that this is a primary role of mathematics, even of mathematical physics. First of all, mathematics provides an abstract language in which one can attempt to state precisely some physical laws. Secondly, mathematics is used as a source of physical concepts. I have always believed in the continued interplay of mathematical and physical ideas. Important physical concepts usually led in the past to an enrichment of the mathematical ideas. Vice versa, a concept which occurs naturally in diverse areas of mathematics must have an important physical interpretation.

We seem to be witnessing a rebirth of the classical attitudes. The fundamental works of Truesdell, Noll, Coleman, and Gurtin in continuum mechanics; the works of Lichnerowicz and Hermann in modern physics, and the unexpected application of the Attiya-Singer index theory to quantum mechanics, the "rigorous" theories of Feynman integration-all point towards a new era of physical interpretation of mathematical concepts. This monograph attempts to contribute in a modest way toward this general trend.

The author realizes how hard it is for an engineer to absorb new mathematical ideas. At the same time, more and more do the modern mathematical ideas filter into the graduate courses of our engineering colleges. Just how much is to be taken for granted is hard to decide. What sufficed in the 1950-s is insufficient in the 1980-s.

As a compromise, some elementary concepts of functional analysis have been included in the Appendices A and B, Volume I, of this work. This monograph covers only a narrow range of mathematical material which is generally labeled "variational methods" trying to bypass most of the typical 19-th century arguments which are the backbone of most "mathematical methods for ..." expository materials and textbooks.

Volume 2 deals with related topics. Algebraic approach (Lie group, Lie algebras), invariance theory; modern theories of sensitivity; connection with problems of control theory and optimization theory.

It is impossible to treat all important aspects of the interaction between physics, engineering and mathematics even when we relate it exclusively to the variational approaches.

The author selected some topics, omitted others (perhaps even more important) and relied on some mathematical developments, while at best paying only a lip service to others. Some developments in the theory of design optimization, sensitivity, group theoretic and model theoretic methods grow so fast that any research monograph is bound to be slightly out of date by the time it is published.

The author has no intention of producing an encyclopedic text, or trying to complete with such texts, but concentrates specifically in Volume 1 of this work on the critical point theory and its applications to continuum mechanics.

CHAPTER 1

ENERGY METHODS, CLASSICAL CALCULUS OF VARIATIONS APPROACH - SELECTED TOPICS AND APPLICATIONS

1.1 The Energy Methods. -- Some Engineering Examples

The points of view of Hamilton, Lagrange, Gauss, Hertz, and Lord Rayleigh emphasized the concept of energy rather than force. The equations of motion of the system are not derived by consideration of equilibrium of forces acting on the system, possibly including the Newtonian and d'Alambert inertia forces. Instead, the primary role is played by energy considerations. As an example of such an approach, a condition of stable equilibrium under static loads is replaced by the condition of a local minimum for the potential energy of a mechanical system. Instead of solving the equations of motion of a vibrating system to find its natural frequencies, it is possible to consider the mean values of potential and kinetic energies, or to minimize an appropriate energy functional.

It was Rayleigh who first discovered that in the natural mode of vibration of an elastic system the equal distribution of average kinetic and potential energies caused the frequency of the system to be minimized. Moreover, any other assumed motion, satisfying the boundary conditions (but not necessarily obeying any physical laws!) will result in a higher value of the fundamental frequency.

This type of a problem is traditionally "solved" by techniques of classical analysis best illustrated by an example of a vibrating string fastened at the ends. Let the string have length ℓ, mass density ρ (per unit length), and be subjected to approximately constant tention T. One could carry out the usual analysis based on Newton's laws of motion to determine the equations of motion and the corresponding natural frequency. As an exercise we shall carry out the details of a heuristic analysis in the general form still offered in many engineering texts.

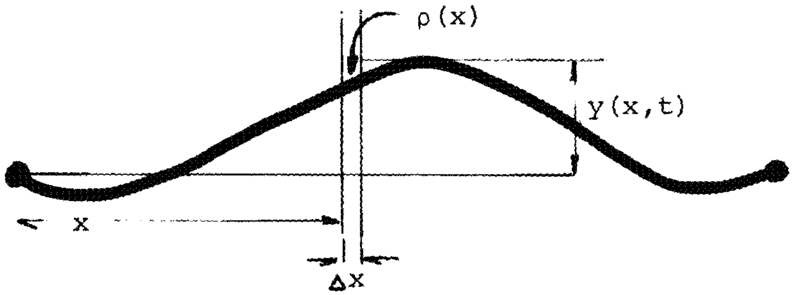

Figure 1

We wish to find the deflection function y(x,t) obeying boundary conditions:

$$y(o) \equiv 0, \quad y(\ell) \equiv 0$$

These boundary conditions are independent of time.

At this point I will repeat a heuristic argument found in many engineering texts, and one which I have heard in my young days in the classroom. We look through a magnifying glass at a small segment of the string which now "looks almost like a segment of a line." (Remark: We have just assumed "spatial differentiability" of the displacement function!)

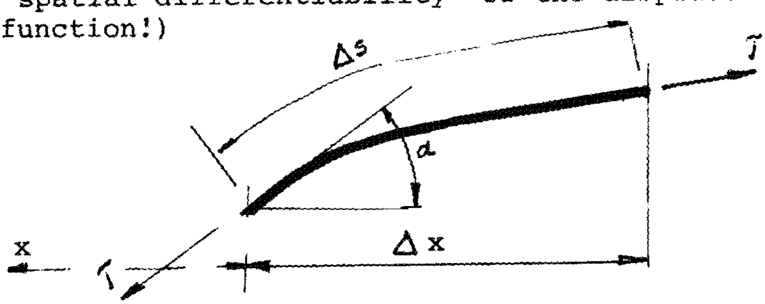

Figure 2

The force of gravity is ignored and we assume that only the tension forces acting on the ends of this segment "are of any importance." Balancing the y components of forces (per unit length of the string), we have

$$[T \sin\alpha/\Delta s]_{x+\Delta x} - [T \sin\alpha/\Delta s]_{x} \cong \rho \frac{\partial^2 y}{\partial t^2} \cdot \Delta x ,$$

where $\cong$ means "is approximately equal."

If α "is small" we substitute $\tan \alpha = \frac{\partial y}{\partial x}$ for $\sin \alpha$, and use the mean value theorem for

$$\frac{[T \frac{\partial y}{\partial x} (1 + (\frac{\partial y}{\partial x})^2]_{\hat{x} + \Delta x} - [T \frac{\partial y}{\partial x} (1 + (\frac{\partial y}{\partial x})^2]_{\hat{x}}}{\Delta x}$$

to write:

$$\rho \frac{\partial^2 y}{\partial t^2}\Big|_{x = \xi} = \frac{\partial}{\partial x}(T \frac{\partial y}{\partial x} (1 + (\frac{\partial y}{\partial x})^2))\Big|_{x = \xi}$$

$x \leqq \xi < x + \Delta x$, and observe that ξ is quite arbitrary. Then we assume that $1 + (\frac{\partial y}{\partial x})^2$ is "very close to unity," and that "T is a constant function of x." Hence, putting $T/\rho = c^2$, we finish by writing the classical equation of the one-dimensional wave propagation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}, \qquad (1.1)$$

describing the dynamic behavior of the string.

Now, let us look at our argument closely to answer a simple question, "what kind of a (genuine physical) string motion is really described by this equation?" To realize what can possibly cause a string not to behave in a fashion which fits into our equation, we should list all the suspicious sounding statements which were quoted in the inverted commas.

Now, assumming that everything is in order, we proceed to separate variables by writing $y(x,t) = X(x) \cdot \theta(t)$. We suggest that the reader should stop here and consider the physical implications of writing $y(x,t) = X(x) \cdot \theta(t)$. What are the physical implications of this mathematical assumption? These are certainly nontrivial. Let us overlook this point, and simply assume that the solution of the problem $y(x,t)$ can be written in the separated form

$$y(x,t) = X(x) \cdot \theta(t) \qquad (1.2)$$

It follows easily that

$$\frac{X''(x)}{X(x)} = \frac{1}{C^2}\frac{\theta''(t)}{\theta(t)} = \text{constant};$$

We denote this constant by $-\Lambda^2$. Then $\theta(t)$ satisfies the equation

$$\theta'' + \omega^2\,\theta = 0\ ,\ \text{where}\ \omega = \Lambda C,$$

and $\theta = A_o \sin(\omega t)$ solves it. ω is interpreted as the natural angular velocity. The constant A_o is the amplitude. The natural frequency is given by $f = \omega/2\pi$.

A more sophisticated approach would consider the average values of potential and kinetic energies.

$$T = \frac{1}{2}\int_o^{\ell} \rho\,\left(\frac{dy}{dt}\right)^2\,dx$$

Suppose that the string is vibrating with frequency f, where $\omega = 2\pi f$. Integrating over a complete cycle of vibration gives a formula for average kinetic energy $\bar{T}$ and potential energy $\bar{V}$.

$$\bar{T} = \frac{1}{4} \rho \, \omega^2 \int_0^{\ell} y^2 dx,$$

$$\bar{V} = \frac{1}{4} E \int_0^{\ell} \left(\frac{dy}{dx}\right)^2 dx,$$

where E denotes Young's modulus.

We replace the constant tension assumption by the equivalent assumption stating that the cross-sectional area of the string is constant and Young's modulus is constant.

Denoting by $\langle f, g \rangle$ the $L_2\,[0,\ell\,]$ product, i.e., $\langle f, g \rangle = \int_0^{\ell} f(x) \cdot g(x)\, dx$, we rewrite the formula for $\omega_1^{\,2}$ in this notation.

$$\omega_1^{\,2} = \frac{E}{\rho} \, \frac{\langle Ay_1, y_1 \rangle}{\langle y_1, y_1 \rangle},$$ where A stands for the operator:

$$A \equiv - \frac{d^2}{dx^2} \,.$$

According to Rayleigh's principle, the natural mode $y_1(x)$ minimizes $\omega_1^{\,2}$. That is, among all possible shapes $\eta(x)$ which are physically admissible, and satisfy $\eta(0) = \eta(\ell) = 0$, the natural mode $y_1(x)$ minimizes $\omega_1^{\,2}$. Hence, $\omega_1^{\,2}$ may be regarded as a functional depending on the shape and distribution of weight of the vibrating string $\eta(x)$, i.e., $\omega_1^{\,2} = \omega_1^{\,2}(\eta)$. One may investigate this dependence and the <u>sensitivity</u> of ω_1.

If we know how to differentiate ω_1^2 with respect to η, the necessary condition for the minimum of ω_1^2 becomes $\frac{d(\omega_1^2)}{d\eta} = 0$. According to the rules of Fréchet differentiation given later in the Appendix A,

$$\frac{d}{d\eta}(\omega_1^2(\eta)) = \frac{d}{d\eta}\left(\frac{\langle A\eta,\eta\rangle}{\langle\eta,\eta\rangle}\right) = \frac{2}{\langle\eta,\eta\rangle}(A\eta - \omega_1^2\eta) = 0$$

This is really the equation for the vibrating system in its fundamental mode determining the best design. The equation of motion is given by $d(\omega_1^2)/dy(x) = 0$, where ω is regarded as a function of the unknown displacement. That is, we claim that the derivative of the Rayleigh quotient with respect to y(x) is equal to zero.

It is interesting to observe that the entire discussion considering equilibrium of forces acting on the string has been bypassed, and replaced by the simple-minded statement that the first derivative must be equal to zero when a differentiable function (functional) assumes a local minimum.

This is a deep observation contrasting the points of view of Newton and Huygens, but also providing an insight into modern viewpoint of classical and continuum mechanics.

The important aspect of our simple example is the replacement of analysis concerning vectors (forces, displacements, etc.) by an analysis concerning some extremal properties of a functional (a function whose range is a subset of real or complex numbers). Specifically, we differentiate the energy functional with respect to some vector. In examples offered in this chapter, this is a straightforward operation. However in some example discussed in later chapters concerning continuum mechanics, it is not clear what "differentiation" means or how to compute such abstract derivatives. One needs some preparation in functional analysis. Elementary concepts of functional analysis as needed in critical point theory are given in Appendices A and B of this volume of the monograph.

First, let us illustrate the energy approach by an elementary example of a direct application of the energy method.

Consider a mass m attached to two points by linear springs and subjected to the force of gravity, as shown on Figure 3.

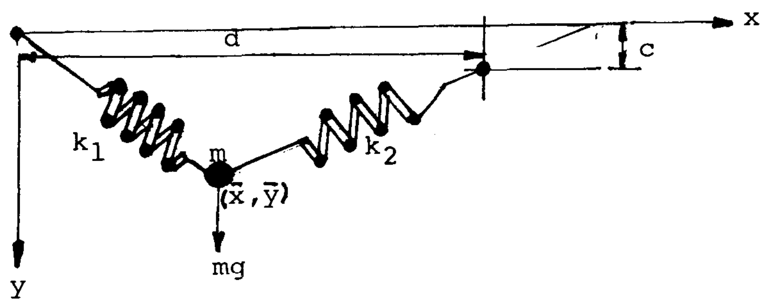

Figure 3

The spring constants are k_1, k_2 as indicated. Find the position of equilibrium.

Instead of attempting to balance the forces and moments acting on the mass m, as is done in an elementary statics course, we shall use the principle of minimum potential energy.

We choose coordinates x, y as indicated on Figure 3 so that the force of gravity acts along a line parallel to axis y. The potential energy is given by

$$V = \frac{1}{2} K_1 (\bar{x}^2 + \bar{y}^2) + \frac{1}{2} K_2 [(d-\bar{x})^2 + (\bar{y}+c)^2] + mg\bar{y}.$$

Note that $\bar{y}$ appears to be negative on the Figure A-1; however the positive y-direction is downward. There is no reason why the positive direction should be up!

V attains a minimum only if

$$\frac{\partial V}{\partial \bar{x}} = \frac{\partial V}{\partial \bar{y}} = 0, \text{ or } \frac{\partial V}{\partial \bar{x}} = K_1\bar{x} - K_2(d-\bar{x}) = 0$$

$$\frac{\partial V}{\partial \bar{y}} = K_1\bar{x} + K_2(\bar{y} + c) + mg = 0 \quad .$$

Solving for $\bar{x}$ and $\bar{y}$, we obtain the coordinates of the equilibrium point.

$$\bar{x} = \frac{K_2 d}{K_1 + K_2} \, , \; \bar{y} = \frac{mg + K_2 c}{K_1 + K_2} \quad .$$

We check that forces acting on the mass m are in balance (i.e., their sum is equal to zero). Indeed they are.

Somewhat similar problems can be regarded as exercises.

1. Compute the equilibrium position of the mass suspended as shown by three linear springs by minimizing the potential energy of this system.

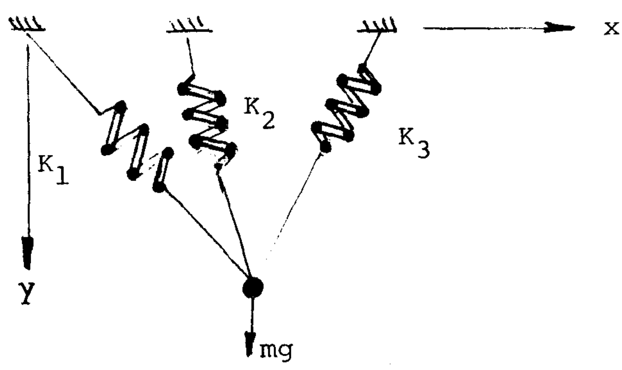

Figure 4

(Observe that this problem is not statically determinate!)

2.

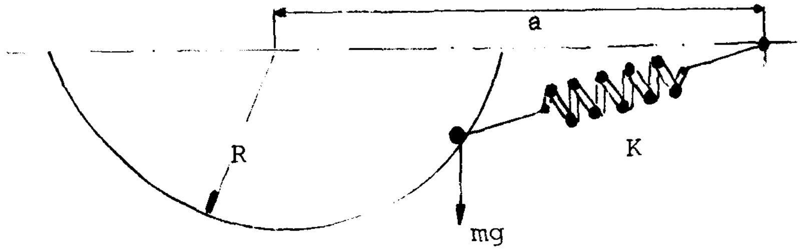

Figure 5

A mass is constrained to a circular path (as shown). A linear spring attracts the mass towards a point A with a > R. Find the equilibrium position.

1.2 Examples From Structural Mechanics

The Theorems of Castigliano and Betti

Suppose that external forces are applied to a structure which deflects elastically. The deflections are "small" and are linearly dependent on the applied forces, i.e., if the forces applied are given by an n-vector, the m-dimensional deflection vector is of the form

$$\underset{\sim}{q} = A\underset{\sim}{f}$$

where A is an m x n matrix. Let us assume Hooke's law. Castigliano's theorem asserts that the deflection δ in the direction of a force f_i at the point of application of that force is equal to the derivative with respect to that force of the total strain energy of the structure:

$$\delta_i = \frac{\partial U}{\partial f_i}$$

This formula leads to direct computational results based on approximate formulas for energy forms.

The approximate formulas for strain energy are given below. The strain energy of a uniform elastic bar in pure tension, or compression, is

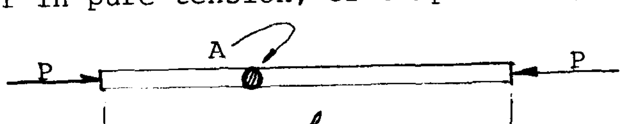

Figure 6

$$U = \frac{P^2 \ell}{2AE},$$

where

P is the force.
A is the cross-sectional area.
E is Young's modulus.
ℓ is the length.

For a pin-jointed structure containing n-members, the analogous formula is

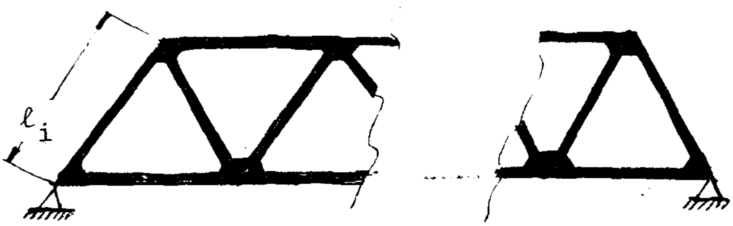

Figure 7

$$U = \sum_{i=1}^{n} \frac{S_i^2 \ell_i}{2EA_i}$$

A_i is the area of cross section of the i-th member.
ℓ_i is the length of the i-th member.

S_i is the force transmitted by the i-th member.

E is the Young's modulus.
For a single beam in bending, the strain energy is

$$U = 1/(2E) \int_o^{\ell} \left(\frac{M^2(x)}{I(x)}\right) dx$$

M(x) is the bending moment.
I(x) is the moment of inertia of the cross-sectional area about the neutral axis.