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PREFACE

This volume contains contributions by some of the invited speakers of the First International Symposium on Algebraic Structures and Number Theory held in Hong Kong during the summer of 1988.

S. P. Lam

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Linearization Problems on Affine Varieties

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Introduction.

We consider affine varieties, and work over $\mathbb{C}$ for convenience of exposition, though many of the issues discussed are of interest over other base fields. The basic linear objects are the affine spaces $\mathbb{C}^n$ and the linear maps between them. In general one linearly approximates varieties locally by their tangent spaces, and morphisms by their derivatives.

We are interested here in some global linearization problems, of the following general types.

I. Linearization of varieties. Find natural criteria for recognizing when a variety is linear (i.e. isomorphic to some $\mathbb{C}^d$).

II. Differential criteria for global properties. Deduce global properties from conditions on tangent bundles or derivatives.

III. Linearization of morphisms. Find natural conditions for morphisms $f : \mathbb{C}^d \to \mathbb{C}^n$ to be linear in suitable global coordinate systems.

IV. Linearization of families. Given a family of such objects parametrized over a base variety S, and assuming that they are linear locally, or perhaps only fiberwise, over S, to what extent is the family globally linear over S?

I propose to survey a number of well known problems that fit into this framework, and to give some information about the present state of knowledge, as well as some background references. It will be apparent that our knowledge is in a very primitive state, though the problems are very naive and natural.

I. LINEARIZATION OF VARIETIES

The basic issue here is to find "useful" characterizations of $\mathbb{C}^d$ as an algebraic variety. Some problems on which to test usefulness are the following.

Cancellation Problem. From the condition $X \times \mathbb{C}^n \cong \mathbb{C}^{d+n}$ can we conclude that $X \cong \mathbb{C}^d$? I.e., are "stably linear varieties" linear?

More generally we have the:

Cartesian Factor Problem. If $X \times Y \cong \mathbb{C}^n$ then is $X \cong \mathbb{C}^d$ ($d = \dim(X)$)?

Reductive Quotient Problem. If a reductive group G acts on $X = \mathbb{C}^m$ and if X/G is smooth of dimension d, then is $X/G \cong \mathbb{C}^d$?

Solving this affirmatively does likewise for the cancellation problem; for example let $GL_n(\mathbb{C})$ act on the second factor of $X \times \mathbb{C}^n \cong \mathbb{C}^{d+n}$.

Let $X = \text{Spec}(A)$ be an affine variety of dimension d. Following are some *necessary* conditions for X to be isomorphic to $\mathbb{C}^d$.

(*a*) A is factorial: $Pic(X) = 0$.
(*b*) A has only constant units: $A^\times = \mathbb{C}^\times$.
(*c*) X is smooth.
(*d*) X is affine 1-ruled. (Explained below.)
(*e*) X has logarithmic Kodaira dimension $\overline{\kappa}(X) = -\infty$ (cf. [**FI**]).
(*f*) X is contractible.
(*g*) X is simply connected at infinity: $\pi_1^\infty(X) = 1$.

For $d = 1$ it is easy to see that: $(a) + (b) \Rightarrow X \cong \mathbb{C}$. For $d = 2$ the known results are much deeper. First we explain that "affine 1-ruled" signifies the following equivalent conditions (cf. [**MS**]):

(*i*) A admits a locally nilpotent derivation $\neq 0$.
(*ii*) X admits a non-trivial action of $\mathbb{G}_a$.
(*iii*) X contains an open subvariety of the form $Z \times \mathbb{C}$.

For $X a$ surface ($\dim(X) = 2$) we have the following results.

(1) (C. P. Ramanujam, [**R**]). $(c) + (f) + (g) \Rightarrow X \cong \mathbb{C}^2$.
(2) (Miyanishi, [**M**]). $(a) + (b) + (c) + (d) \Rightarrow X \cong \mathbb{C}^2$.
(3) (Fujita, [**F**]). $(c) + (e) \Rightarrow (d)$. Hence, $(a) + (b) + (c) + (e) \Rightarrow X \cong \mathbb{C}^2$.

Now it was observed by Fujita and Iitaka [**FI**] that condition (*e*) is cancellation invariant:

(4) $X \times V \cong \mathbb{C}^d \times V \Rightarrow \overline{\kappa}(X) = -\infty$.

Thus (3) and (4) solve the Cancellation Problem in dimension two:

(5) (Fujita, [**F**]). $X \times \mathbb{C}^n \cong \mathbb{C}^{n+2} \Rightarrow X \cong \mathbb{C}^2$.

Fujita also formulates the:

CONJECTURE (Fujita, [**F**]). *If* $\dim(X) = d$ *then*

$$(c) + (e) + (f) \Rightarrow X \cong \mathbb{C}^d.$$

If true, this Conjecture would solve the Cancellation Problem in all dimensions.

For $d \geq 3$ very little is known on these questions. Miyanishi and Sugie (unpublished) have announced that $X \cong \mathbb{C}^3$ under the assumptions that $X \times \mathbb{C}^n \cong \mathbb{C}^{n+3}$, X is affine 1-ruled, and a further technical condition related to the $\mathbb{G}_a$-action on X.

II. Differential criteria for global properties

The archetypal problem here is the:

JACOBIAN CONJECTURE. *If $F : \mathbb{C}^d \to \mathbb{C}^d$ has everywhere invertible derivative $F'(x)$ (i.e. the Jacobian determinant $\det F'(x)$ is a constant $\neq 0$) then F is invertible.*

This is trivial for $d = 1$, and remains an open problem for all $d \geq 2$ (cf. [**BCW1**]).

Immersion Problem. Suppose that $X = \mathrm{Spec}(A)$ is parallelizable (i.e. has trivial tangent bundle) of dimension d. Under what conditions can we immerse X in $\mathbb{C}^d$?

The parallelizability condition means that $\Omega_{A/\mathbb{C}} \cong A^d$. Then the immersion condition asks for an A-basis of $\Omega_{A/\mathbb{C}}$ of the form $df_1, \ldots, df_d$ for suitable $f_i \in A$.

Here is the kind of situation where we would like to apply this. Suppose that $X \times \mathbb{C}^n \cong \mathbb{C}^{d+n}$. Then it follows easily from the Quillen–Suslin Theorem (cf. [L]) that projective A-modules are free, hence X is parallelizable. Suppose that we could then further conclude that X admits an immersion $f : X \to \mathbb{C}^d$. Then $F =: f \times Id : X \times \mathbb{C}^n \to \mathbb{C}^d \times \mathbb{C}^n$ can be interpreted as an immersion $\mathbb{C}^{d+n} \to \mathbb{C}^{d+n}$. Assuming the Jacobian Conjecture, we would conclude that F is an isomorphism, hence so also is f. Thus we have an interesting logical dependence among the Cancellation Problem, the Jacobian Conjecture, and the Immersion Problem for stably linear varieties.

III. Linearization of morphisms

Embedding Problem. Is every closed embedding $\mathbb{C}^d \to \mathbb{C}^{d+m}$ linearizable, i.e. linear relative to suitable global coordinates in $\mathbb{C}^{d+m}$?

Let us formulate this more concretely in the algebraic setting. We write $R^{[n]}$ for a polynomial algebra in n variables over a ring R. The embedding above thus corresponds to a surjection $f : \mathbb{C}^{[d+m]} \to \mathbb{C}^{[d]}$, say with kernel J. Then linearization asks for a system of variables $x_1, \ldots, x_{m+d}$ of $\mathbb{C}^{[d+m]}$ such that (i) $x_1, \ldots, x_m$ generate the ideal J, and (ii) $f(x_{m+1}), \ldots, f(x_{m+d})$ generate the algebra $\mathbb{C}^{[d]}$. In fact (i) implies (ii), and, conversely, given (ii), we can modify $x_1, \ldots, x_m$ by polynomials in $x_{m+1}, \ldots, x_{m+d}$ to achieve (i). Note that linerization includes the condition that $\mathbb{C}^d$ is a "complete intersection" in $\mathbb{C}^{d+m}$, i.e. that J is generated by m elements.

A few cases of the Embedding Problem are trivially affirmative. When $d = 0$, every embedding of a point in $\mathbb{C}^m$ is obviously linearizable. When $m = 0$, every embedding $\mathbb{C}^d \to \mathbb{C}^d$ is easily seen to be an isomorphism, hence linearizable.

For $m = 1$ an embedding $\mathbb{C}^d \to \mathbb{C}^{d+1}$ is defined by one equation, $t = 0$. We ask whether $\mathbb{C}^{[d+1]}t = \mathbb{C}^{[d+1]}x_0$ for some system $x_0, \ldots, x_d$ of variables for $\mathbb{C}^{[d+1]}$. Then t, being a scalar multiple of x_0, is itself a variable.

This case ($m = 1$) of the Embedding Problem is already highly non-trivial. It includes the only known substantial results. In the case $m = d = 1$, the Abhyankar–Moh Embedding Theorem [**AM**] linearizes closed embeddings of the line in the plane. Russell and Sathaye [**RS**] have linearized special cases of embeddings of $\mathbb{C}^2$ in $\mathbb{C}^3$.

We saw that the case $m = 1$ of the Embedding Problem is the case $m = 1$ of the following:

Variable Recognition Problem. How can we recognize when $t = (t_1, \ldots, t_m)$ in $A = \mathbb{C}^{[m+d]}$ is part of a system of variables for A?

A necessary condition is that $dt_1, \ldots, dt_m$ be a basis for a (free) direct summand of $\Omega_{A/\mathbb{C}}$. That this condition should also suffice when $d = 0$ is just the Jacobian Conjecture. However it is definitely not sufficient for $d > 0$. Consider for example the case $d = m = 1$; take $t = x + (xy)^2 \in A = \mathbb{C}[x, y]$. Then $(1 - 2xy^2)t_x + 2y^3 t_y = 1$, so $dt = t_x dx + t_y dy$ is unimodular in $\Omega_{A/\mathbb{C}}$. However $t = x(1 + xy^2)$, being reducible, cannot be a variable.

For the Variable Recognition Problem it is more natural to consider A as an algebra over $R = \mathbb{C}[t] = \mathbb{C}[t_1, \ldots, t_m]$. To say that t is part of a system of variables for A is to say that $A = R^{[d]}$, a polynomial R-algebra. A necessary condition for this is that the fibers be so:

(a) $A_p/pA_p \cong (R_p/pR_p)^{[d]}$ for all $p \in \operatorname{Spec}(R)$.

The special case of this when $p = (t)R \quad (= Rt_1 + \cdots + Rt_m)$ is:

(a_0) $A/(t)A \cong \mathbb{C}^{[d]}$.

Thus (a_0) says that "$t = 0$" defines an embedding of $\mathbb{C}^d$ in $\mathbb{C}^{d+m}$. If the Embedding Problem is affirmed then we can at least conclude that there is a system of variables $x_1, \ldots, x_{m+d}$ for A so that $t_1, \ldots, t_m$ generate the same ideal of A as $x_1, \ldots, x_m$. When $m = 1$ this is, as observed above, equivalent to t_1 being a variable.

However for $m \geq 2$ (a_0) no longer suffices to make $(t_1, \ldots, t_m)$ part of a system of variables for A. For example ($m = d = 2$) in $A = \mathbb{C}[x, y]$ take $t_1 = xy - 1$ and $t_2 = x - 1$. Then $t_1 = 0$, $t_2 = 0$ defines the (reduced) point $(1, 1) \in \mathbb{C}^2$, yet t_1 is not a variable: $A/t_1 A = \mathbb{C}[x, x^{-1}]$. This shows that something stronger than (a_0), e.g. (a), is necessary.

A further natural condition to require is:

(b) A is a flat R-module.

(This is automatic when $m = 1$.) It seems reasonable to ask whether (a) plus (b) imply that $A = R^{[d]}$. For $m = 1$ and $d = 2$ this follows from a result of Sathaye [S], using [BCW2]. In the general case it has been shown by Asanuma ([A], cf. also [BS]) that $A^{[q]} \cong R^{[d+q]}$ (R-algebra isomorphism) for some $q \geq 0$. Asanuma further shows that, in characteristic $p > 0$, one may need $q > 0$ here.

Further discussion of this problem is given below in section IV, Linearization of families.

Automorphisms. The group

$$GA_n(\mathbb{C})$$

of polynomial automorphisms of $\mathbb{C}^n$, is called the *affine Cremona group*. It contains the groups of *linear* automorphisms, $GL_n(\mathbb{C})$, of *translations*, $\cong \mathbb{C}^n$, and of *affine linear* automorphisms,

$$Af_n(\mathbb{C}) \cong GL_n(\mathbb{C}) \ltimes \mathbb{C}^n.$$

For $n = 1$ we have $GA_1(\mathbb{C}) = Af_1(\mathbb{C})$. For $n \geq 2$ there are many essentially non-linear automorphisms. A natural source is the (non-linear) *triangular group*, $BA_n(\mathbb{C})$, consisting of automorphisms of the form

$$(x_1, \ldots, x_n) \mapsto (x'_1, \ldots, x'_n)$$
$$x'_i = a_i x_i + f_i(x_1, \ldots, x_{i-1})$$

where $a_i \in \mathbb{C}^\times$ and $f_i \in \mathbb{C}[x_1, \ldots, x_{i-1}]$ $(i = 1, \ldots, n)$. Though $BA_n(\mathbb{C})$, unlike $Af_n(\mathbb{C})$, is infinite dimensional, it is a solvable group and reasonably tractable for calculation. Its elements are sometimes called Jonquieres transformations.

The Generation Gap. Is $GA_n(\mathbb{C})$ generated by $Af_n(\mathbb{C})$ and $BA_n(\mathbb{C})$?

For $n = 2$ we have an affirmative answer.

THEOREM (Jung–Van der Kulk). *$GA_2(\mathbb{C})$ is the amalgamated free product of its subgroups $Af_2(\mathbb{C})$ and $BA_2(\mathbb{C})$.*

Actually this is Jung's theorem [J]; Van der Kulk proved the analogue in characteristic p [VdK]. For a proof in the above form see [N] or [W].

For $n \geq 3$ the Generation Gap remains an open problem; some experts suspect a negative answer. On the other hand Marilena Pittaluga has affirmed Lie algebra [Pi1] and topological [Pi2] analogues of the question.

Group Actions. Let G be an algebraic group acting (algebraically) on $\mathbb{C}^n$. Such an action corresponds to an ("algebraic") homomorphism

$$\rho : G \to GA_n(\mathbb{C}),$$

a "non-linear representation". Linearizing such an action means finding global coordinates on $\mathbb{C}^n$ relative to which G acts linearly; or, equivalently, linearizing means conjugating $\rho(G)$ (in $GA_n(\mathbb{C})$) into $GL_n(\mathbb{C})$. A first requirement for this is that G have a fixed point, i.e. that $\rho(G)$ be conjugate to a subgroup of $GA_n^0(\mathbb{C})$, the stabilizer of the origin. We have

$$GA_n^0(\mathbb{C}) = GL_n(\mathbb{C}) \ltimes GA_n^1(\mathbb{C})$$

where $f \in GA_n^1(\mathbb{C})$ iff $f(0) = 0$ and $f'(0) = Id$.

Translations, lacking fixed points, can't be linearized. In fact there are simple actions of $\mathbb{G}_a$ which cannot even be made affine linear. For example, given any $f(x) \in \mathbb{C}[x]$, $\mathbb{G}_a$ acts on the plane by $t : (x, y) \mapsto (x, y + tf(x))$. In fact it can be deduced from the Jung–Van der Kulk Theorem that every action of $\mathbb{G}_a$ on $\mathbb{C}^2$ is conjugate to one of this type. This fact tempts one to conjecture perhaps that actions of unipotent groups on $\mathbb{C}^n$ can be triangularized, i.e. conjugated into $BA_n(\mathbb{C})$. This would be a kind of non-linear Lie–Kolchin Theorem. However such is not the case; in **[B1]** one finds a non-triangularizable action of $\mathbb{G}_a$ on $\mathbb{C}^3$.

Every group with non-trivial unipotent radical admits a non-linearizable action on some $\mathbb{C}^n$. On the other hand we have the:

KAMBAYASHI CONJECTURE (**[K]**). *Every action of a reductive group on $\mathbb{C}^n$ can be linearized.*

Though many experts are skeptical about this conjecture, it is now the focus of much active research, and there are many partial results which support it; we discuss some of them below. First a remark to put some perspective on the conjecture.

REMARK: Suppose that for some group $G \neq \{1\}$ we can linearize all actions of G on affine spaces. Then the Cancellation Theorem holds (in all dimensions). For suppose that $X \times \mathbb{C}^n \cong \mathbb{C}^{d+n}$. After increasing n if necessary we can assume that G admits a linear action on $\mathbb{C}^n$ so that $(\mathbb{C}^n)^G = \{0\}$. (For example use a direct sum of copies of a non-trivial irreducible G-representation.) Then let G act on $X \times \mathbb{C}^n$ using the trivial action on X. We then have $(X \times \mathbb{C}^n)^G = X \times \{0\} \cong X$. On the other hand, interpreting this as a G-action on $\mathbb{C}^{d+n}$, it is, by assumption, linearizable, so its fixed points form a linear variety, whence X is linear.

It is thus tempting to try to obtain Cancellation by showing that one can linearize affine space actions of say $G = \mathbb{G}_m$ (if one likes tori), or $G = \{\pm 1\}$ if one likes finite groups, or $G = SL_2(\mathbb{C})$ if one likes connected semi-simple groups.

Following is a partial list of recent results (cf. **[B2]** and the references there). We consider a reductive group G acting on $\mathbb{C}^n$.

Fixed points exist in the following cases.

(*a*) G is a finite p-group. (Shafarevich, cf. **[B-B]**, or Smith Theory). (See **[PR]** for more general finite groups.)

(b) G is a torus (Bialynicki-Birula [B-B]).
(c) $G = SL_2(\mathbb{C})$ and there are no 3-dimensional orbits (Panyushev [P]).
(d) $G = SL_2(\mathbb{C})$ and $n \leq 7$ (Kraft).
(e) $\dim(\mathbb{C}^n/G) \leq 1$ (Kraft–Luna).

Linearization is possible in the following cases.

(a) G is a torus with an orbit of codimension ≤ 1 (Bialynicki–Birula [B-B]).
(b) G is connected semi-simple and $n \leq 4$ ($n = 1$: trivial; $n = 2$: Corollary of Jung–Van der Kulk Theorem; $n = 3$: Kraft–Popov [KP]; $n = 4$: Panyushev [P]).
(c) If every closed orbit is a point then $\mathbb{C}^n = (\mathbb{C}^n)^G \times V$ where V is a linear representation ([BH]). The action of G is linearizable iff the stably linear variety $(\mathbb{C}^n)^G$ is linear (Cancellation), e.g. if $\dim(\mathbb{C}^{nG}) \leq 2$ (Fujita's Theorem).
(d) An action $F : G \times \mathbb{C}^n \to \mathbb{C}^n$ is given by an n-tuple $(F_1, \ldots, F_n)$ of polynomials in n variables with coefficients in the affine algebra of G. Thus we can define the *polynomial degree*, $\deg(F) = \max_i \deg(F_i)$ of the action. Jurkiewicz [JJ] has shown that if G is a diagonalizable group then all quadratic actions ($\deg(F) \leq 2$) of G on $\mathbb{C}^n$ are linearizable.

The first case left open by (a) concerns $\mathbb{G}_m$ actions on $\mathbb{C}^3$. Kraft has gone quite far toward proving that these actions are linearizable. Kraft and Schwartz have also obtained strong information on actions for which $\dim(\mathbb{C}^n/G) \leq 1$.

G-Varieties. Let G be an algebraic group. By a G-variety we understand an (affine) variety X equipped with a G-action. We call it a *linear G-variety* if X is G-isomorphic to some $\mathbb{C}^n$ with a linear G-action, and *stably linear* if $X \times Y$ is linear for some linear G-variety Y. The "Equivariant Cancellation Problem" asks whether stably linear G-varieties are linear. Similarly one can formulate equivarient versions of essentially all of the problems discussed above.

Algebraically, a G-variety corresponds to an affine $\mathbb{C}$-algebra A on which G acts (rationally) as algebra automorphisms. We call A a *$\mathbb{C} - G$-algebra.* If $A \to B$ is a morphism of $\mathbb{C} - G$-algebras we call B an *$A - G$-algebra.* There are various senses in which B might be "linear" over A.

For any finite dimensional (rational) $\mathbb{C} - G$-module V put $S_A(V) = A \otimes_{\mathbb{C}} \mathrm{Sym}_{\mathbb{C}}(V)$. Such an $A - G$-algebra is said to be *linear*. An $A - G$-algebra B is *stably linear* if $B \otimes_A C$ is linear for some linear $A - G$-algebra C. We call B a *vector bundle $A - G$-algebra* if $B \cong \mathrm{Sym}_A(P)$ for some finitely generated projective $A - G$-module P.

IV. Linearization of families

Suppose that A (and B) are $R-G$-algebras where R is a $\mathbb{C}-G$-algebra with trivial G-action. Then for each $p \in \operatorname{Spec}(R)$ we have the *localized morphism* $A_p \to B_p$ of $R_p - G$-algebras, and the *fiber morphism* $A(p) \to B(p)$ of $R(p)-G$-algebras, where $R(p) = R_p/pR_p$, $A(p) = R(p) \otimes_R A$, etc. We ask here whether linearity properties locally, or fiberwise, imply global linearity properties.

For localization we have the best we could hope for.

THEOREM (**[BS]**). *If, for all $p \in \operatorname{Spec}(R)$, B_p is a vector bundle $A_p - G$-algebra then B is a vector bundle $A - G$-algebra.*

This is proved by the methods of **[BCW2]**, which treats the non-equivariant case.

Effective treatment of algebras with linear fibers was pioneered, in the non-equivariant case, by Asanuma **[A]**. Asanuma's methods and results are generalized to the equivariant setting in **[BS]**. The following concept emerges as being central. We call B an *Asanuma $A-G$-algebra* if there is a finitely generated projective $B-G$-module P such that $\operatorname{Sym}_B(P)$ is a linear $A-G$-algebra. This and the preceding notions are related as follows:

$$\begin{array}{ccc} \text{linear} & \Rightarrow & \text{stably linear} \\ \Downarrow & & \Downarrow \\ \text{vector bundle} & \Rightarrow & \text{Asanuma.} \end{array}$$

Moreover if A is a (stably) linear $\mathbb{C}-G$-algebra, or even a retract of a linear $\mathbb{C}-G$-algebra, then: Asanuma $\Leftrightarrow$ stably vector bundle $\Leftrightarrow$ stably linear.

THEOREM (**[BS]**). *Let $A \to B$ be a morphism of affine $R-G$-algebras. Assume that B is A-flat. Then B is an Asanuma $A-G$-algebra iff, for all $p \in \operatorname{Spec}(R)$, $B(p)$ is an Asanuma $A(p)-G$-algebra.*

Assuming that B is A-flat with linear fibres, we might optimistically want to conclude that B is actually linear, not just Asanuma. First there is a tangent bundle obstruction to this; when A is not regular this obstruction can be non-trivial and even forbid B from being stably a vector bundle $A-G$-algebra. When the tangent bundle obstruction is trivial (say B is "$A-G$-parallelizable") then there are still examples of Asanuma **[A]** in characteristic $p > 0$ to show that B is only stably linear, not linear. Such examples are not known in characteristic 0. Moreover Sathaye **[S]** has shown (in characteristic 0) that a flat family of planes over a PID is globally a plane.

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Valuations on Rings and Simple Algebras

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Rings that can be considered as noncommutative valuation rings occur as coordinate rings of Hjelmslev planes ([17]), in the construction of division rings that are not crossed products ([1], [16]) or in which ordered groups ([23]) or enveloping algebras ([9]) can be embedded, or as components in structure theorems for certain rings ([24]). Noncommutative valuation rings were also used in the investigation of the reduced Whitehead group $SK_1(A)$ of a division algebra A ([25]).

Here we want to discuss three aspects of noncommutative valuation theory. In the first section we consider the extension problem, then in a second part report on some results about the structure of complete discrete rank one valuation rings and higher derivations, and finally we consider ordered groups associated with noncommutative valuation rings and the semigroups of values for right invariant right chain domains.

1. The extension problem.

A valuation v on a commutative field K is a mapping from $K^* = K\backslash\{0\}$ onto an ordered group G that satisfies the following two conditions:

i) $v(ab) = v(a) \cdot v(b)$ and

ii) $v(a+b) \geq \min\{v(a), v(b)\}$ for any $a, b \in K^*$.

If $F \supset K$ is any extension field of K then this mapping v can be extended to a valuation of F. This also means that there exists a valuation ring B of F with $B \cap K = V$, the valuation ring $V = \{a \in K \mid v(a) \geq v(1)\}$ of v in K. Here, a subring B of F is called a valuation ring of F if $a \in F\backslash B$ implies $a^{-1} \in B$, and we say that such a ring B is an extension of V in F if $B \cap K = V$.

If, in addition, F is algebraic over K then the intersection of all extensions of V in F is equal to the integral closure of V in F and if F is Galois over K then $\{\sigma(B) \mid \sigma \in G(F \mid K)\}$ is exactly the set of extensions of V in F, where B is one of these extensions and $G(F \mid K)$ is the Galois group of F over K ([15]).

Replacing in the above discussion the fields by division rings one obtains immediately the notion of a valuation on a division ring, say D, and observes that the corresponding valuation ring $B = \{a \in D^* \mid v(a) \geq 1\}$ satisfies the following two conditions:

T) $x \in D\backslash B$ implies $x^{-1} \in B$, i.e., B is a total subring of D.

I) $dBd^{-1} = B$ for all $0 \neq d \in D$; i.e., B is an invariant subring of D.

Conversely, every total invariant subring B of D defines an ordered group $G = \{aB \mid 0 \neq a \in D\}$ with $aBbB = abB$ as operation and $aB \leq bB$ if and only if $aB \supseteq bB$ as order. The corresponding valuation v is then simply given by $v(a) = aB$.

However, the commutative results cited above do not carry over: If $\mathbb{Z}_p$ is the valuation ring of the p-adic valuation in the rationals $\mathbb{Q}$ and $D = H$ is the division ring of quarternions over $\mathbb{Q}$ then there does not exist an invariant total subring B of H with $B \cap \mathbb{Q} = \mathbb{Z}_p$ for p a prime $\neq 2$. Otherwise, $B/J(B)$ is a division algebra finite dimensional over $\mathbb{Z}_p$, hence commutative. But B contains i and $j \in H$ with $ij = -ji \neq ji$ modulo $J(B)$, where $J(B)$ is the maximal ideal in B. We observe that the last argument also showed that $\mathbb{Z}_p$ has no total extension in H.

The following result describes the condition under which a valuation ring V in the center has an invariant total extension ([30]).

THEOREM 1. *A valuation ring V of the center K of a finite dimensional division algebra D has an invariant total extension B in D if and only if B has a unique extension in every subfield F with $K \subseteq F \subseteq D$.*

It follows that a valuation ring V in K has either no invariant total extension or exactly one. The last case occurs for example if we consider the 2-adic valuation of $\mathbb{Q}$ which can be extended to H or in the case where V is a rank 1 complete valuation of K.

Again, let D be a division algebra of dimension n^2 over its center K and let V be a valuation ring of K. With $\mathcal{B} = \{B \mid B \text{ total in } D \text{ and } B \cap K = V\}$ we denote the set of all total extensions of V in D. If V has rank 1 then every total extension is also invariant. However, the following example shows that V can have non-invariant total extensions if the rank of V is greater or equal to two.

Let V_1, V_2 be the two extensions in $\mathbb{Q}(i)$ of $\mathbb{Z}_5$, the valuation ring of the 5-adic valuation in $\mathbb{Q}$, the field of rationals. Let σ be the nontrivial automorphism of $\mathbb{Q}(i)$, i.e., conjugation, and denote by $R = \mathbb{Q}(i)[[t,\sigma]] = \{\sum_0^\infty t^n a_n \mid a_n \in \mathbb{Q}(i)\}$ the skew power series ring with coefficients in $\mathbb{Q}(i)$ and $at = ta^\sigma$ defining the multiplication, $a \in \mathbb{Q}(i)$. Then D, the skew field of quotients of R, is the ring of skew Laurent series and $K = \mathbb{Q}((t^2))$ is the center of D with $[D : K] = 4$. The power series ring $W = \mathbb{Q}[[t^2]]$ contains the subring $V = \{\sum q_i t^{2i} \in W,\ q_0 \in \mathbb{Z}_5\}$ which has the two extensions $B_i = \{\sum_0^\infty t^n a_n \in R \mid a_0 \in V_i\}$, $i = 1, 2$. The rings B_i are total subrings of D with $B_i \cap K = V$, but $tB_1t^{-1} = B_2$, i.e., they are not invariant.

One can prove the following results about the set $\mathcal{B}$ ([5]):

THEOREM 2. *Let V be a valuation ring of K, the center of the division algebra D with $[D : K] = n^2$. Then any two total extensions of V in D are conjugate in D and their number is at most n.*

Essential for the proof of this result is the existence of an invariant total subring R of D which contains all the total extensions B_i of V and is minimal with this property and $\neq D$