14219

DISTRIBUTIONS AND FOURIER TRANSFORMS

WILLIAM F. DONOGHUE, JR.

DISTRIBUTIONS
AND
FOURIER TRANSFORMS

WILLIAM F. DONOGHUE, JR.
Department of Mathematics
University of California
Irvine, California

ACADEMIC PRESS New York San Francisco London 1969
A Subsidiary of Harcourt Brace Jovanovich, Publishers

COPYRIGHT © 1969, BY ACADEMIC PRESS, INC.
ALL RIGHTS RESERVED
NO PART OF THIS BOOK MAY BE REPRODUCED IN ANY FORM,
BY PHOTOSTAT, MICROFILM, OR ANY OTHER MEANS, WITHOUT
WRITTEN PERMISSION FROM THE PUBLISHERS.

ACADEMIC PRESS, INC.
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS, INC. (LONDON) LTD.
24/28 Oval Road, London NW1

LIBRARY OF CONGRESS CATALOG CARD NUMBER: 69-12285

PRINTED IN THE UNITED STATES OF AMERICA

Preface

In this book I try to give a readable introduction to the modern theory of the Fourier transform and to show some interesting applications of that theory in higher analysis. The book is directed to students having only a moderate preparation in real and complex analysis. More exactly, I suppose the reader to be familiar with the elements of real variables and Lebesgue integration and to have some knowledge of analytic functions. Further along in the book both Hilbert spaces and L^p-spaces play a role, but the reader is presumed to know only a little about either topic, much less, in fact, than appears in any standard modern real variable textbook.

Much of the material the student is expected to know is reviewed in the first part of the book, which also serves to establish our conventions of notation and terminology. Some topics from advanced calculus and analytic function theory are treated here. There have also been adjoined brief discussions of linear topological spaces, analytic functions of several variables, as well as certain aspects of convexity; these subjects are perhaps not strictly needed for the study of the Fourier transform as we undertake it.

Not everything in Part I is needed for the study of Part II which presents the theory of distributions on the n-dimensional real space as well as the theory of the Fourier transform for temperate distributions. The machinery developed in Part II makes it possible to obtain significant results in harmonic analysis in a fairly simple and direct way; this is done in Part III. The whole book can be covered conveniently in a one-year course if one or two special topics in the third part are omitted.

Much of the book closely follows the lectures in harmonic analysis given by L. Hörmander at Stockholm University during the academic year 1958–1959. However, a number of topics covered in those lectures have been omitted, while a good deal of potential theory and analytic function theory has been adjoined; it would be surprising if Professor Hörmander cared to acknowledge the result as his own. Nevertheless, almost everything in this book has been taught me by L. Hörmander and N. Aronszajn.

There are certain inconsistencies in the presentation. To make the book accessible to as wide a readership as possible I have avoided the treatment of distributions on manifolds and never refer to an exterior differential form. This has made it desirable to accept the Green's formula without proof, although it is only needed here for spheres. Sometimes a theorem is proved with the tacit assumption that the functions or linear spaces occurring in the argument are all real, and later that theorem is invoked in a context where the scalars are complex. This abuse is preferred to the repetition of some incantation assuring the reader that the arguments may be modified to cover the case of complex scalars. I have tried to make the notations as traditional and natural as possible, but have not been able to avoid some trivial ambiguities. Thus, for example, a system of points in R^n is generally written x_k, although the same notation is used for the coordinate functions themselves.

A book covering such a wide range of material is bound to contain mistakes. These, I think, are unimportant, so long as the book conveys the mathematical spirit of the apostolic, nay, the Petrine succession, extending from Gauss, Riemann, and Dirichlet, through Hilbert, Courant, Friedrichs, and John.

March, 1969 WILLIAM F. DONOGHUE, JR.

此为试读,需要完整PDF请访问: www.ertongbook.com

Contents

PART I

INTRODUCTION

1. Equicontinuous Families

Let X be a metric space with the metric $d(x, y)$. If $f(x)$ is a uniformly continuous real or complex-valued function defined on X, its *modulus of continuity* is the function

$$\omega(t) = \sup|f(x) - f(y)|, \quad d(x, y) \leqq t,$$

which is defined for all positive t. $\omega(t)$ is monotone nondecreasing in t and approaches 0 as t does. Since it is not always necessary to operate with *the* modulus of continuity of a function, we will say that any function $\omega^*(t)$ which is monotone and vanishes as t approaches 0 and which satisfies the inequality $\omega(t) \leqq \omega^*(t)$ is a modulus of continuity for $f(x)$. The function $f(x)$ is Lipschitzian with Lipschitz constant M if $\omega^*(t) = Mt$ is a modulus of continuity for it; it is Lipschitzian of order α if a function of the form $\omega^*(t) = Mt^\alpha$ will serve as a modulus of continuity. In practice, only values of α which are smaller than 1 are of interest; one easily shows, for example, that on the real line a function Lipschitzian of order α for $\alpha > 1$ is necessarily a constant.

Let $\mathscr{F}$ be a family of functions defined on the metric space X; the family is called *equicontinuous* if there exists a fixed modulus of continuity $\omega(t)$ which serves for all functions in the family.

Theorem (Ascoli–Arzela): Let $\mathscr{F}$ be an infinite equicontinuous family of functions on the compact metric space K which is uniformly bounded, that is, $|f(x)| \leqq M$ for all x in K and all f in $\mathscr{F}$; then $\mathscr{F}$ contains an infinite sequence $f_n(x)$ which converges uniformly on K.

PROOF: The proof is essentially that of the Bolzano–Weierstrass theorem. For any small positive ε, there exists a finite set $F = [x_1, x_2, \ldots, x_n]$ of points of K such that every point of K is in an ε-neighborhood of at least one point in F. (This is just the assertion that the compact K is totally bounded.) For the same ε, we divide the circle $|z| \leqq M$ into l disjoint sets G_j of diameter at most ε. Next, we partition the family $\mathscr{F}$ into l^n disjoint subfamilies; each subfamily being described by l^n assertions of the form $f(x_i)$ in G_j. Since $\mathscr{F}$ is infinite, at least one of the subfamilies is also infinite, and for any two functions $f_1(x)$ and $f_2(x)$ belonging to the same subfamily, we have

$$|f_1(x) - f_2(x)| \leqq |f_1(x) - f_1(x_i)| + |f_1(x_i) - f_2(x_i)| + |f_2(x_i) - f_2(x)|.$$

If we choose x_i in F so that $d(x_i, x) < \varepsilon$, the first and last terms are bounded by $\omega(\varepsilon)$; the middle term is bounded by ε since both numbers $f_1(x_i)$ and $f_2(x_i)$

belong to the same set G_i; thus, independently of x,

$$|f_1(x) - f_2(x)| \leqq 2\omega(\varepsilon) + \varepsilon .$$

Accordingly, for a fixed small ε, we find an infinite subfamily $\mathscr{F}_1$ of $\mathscr{F}$ having the property that any two functions in $\mathscr{F}_1$ differ by at most $2\omega(\varepsilon) + \varepsilon$ anywhere in K. Passing to $\varepsilon/2$ and arguing with the family $\mathscr{F}_1$, we obtain an infinite subfamily associated with the bound $2\omega(\varepsilon/2) + (\varepsilon/2)$, and continuing in this fashion, we obtain an infinite descending sequence of subfamilies $\mathscr{F}_n$ associated with the bounds $2\omega(\varepsilon 2^{-n}) + 2^{-n}\varepsilon$ which converges to 0. We have now only to choose $f_n(x)$ in the family $\mathscr{F}_n$ distinct from the previous $f_k(x)$ in order to obtain an infinite sequence converging uniformly on K.

The sequence $f_n(x)$ obviously converges to a continuous limit $f^*(x)$ which in general does not belong to the family $\mathscr{F}$; however, the function $\omega(t)$ is a modulus of continuity for f^* and $|f^*(x)| \leqq M$ on K.

When the metric space X is the union of a sequence of compact sets and the family of functions $\mathscr{F}$ is uniformly bounded and equicontinuous on each compact, we can evidently extract a subsequence which converges uniformly on all compact subsets of X.

An important special case of the foregoing arises in function theory. We suppose that $\mathscr{F}$ is a family of functions analytic in some region G and uniformly bounded there by M; if K is a compact subset of G, it can be surrounded by a rectifiable curve C lying wholly in G. We let d denote the distance from K to the curve C, and note that for any $f(z)$ in the family and any z in K

$$|f'(z)| = \left| \frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{(z - \zeta)^2} \right|$$
$$\leqq \frac{ML}{d^2},$$

where L is the length of the curve C. Thus the derivatives of functions in $\mathscr{F}$ are uniformly bounded on K and hence those functions are all Lipschitzian with the same Lipschitz constant, that is, the family is equicontinuous on K. Accordingly, when the family is infinite, we can extract an infinite sequence which converges uniformly on all compact subsets of G to a limit which is also analytic in G and bounded there by M.

We apply this remark to prove the following theorem which may easily be generalized.

Theorem: Let $f(z)$ be analytic and bounded in G, the sector $0 < |z| < R$, $|\arg z| < c$, and suppose $f(x)$ approaches 0 as the real x does; then $f(z)$ converges to 0 uniformly in the sector $|\arg z| \leqq d$ for any $d < c$.

PROOF: (See Fig. 1) We suppose $R \geqq 2$ and consider the compact K defined by $\frac{1}{2} \leqq |z| \leqq 1$ and $|\arg z| \leqq d$ as well as the sequence of functions $f_n(z) = f(2^{-n}z)$ which is uniformly bounded in G and hence equicontinuous on K. We may extract a subsequence converging uniformly on K to an analytic limit $f^*(z)$. On the intersection of the real axis with K, we have $f_n(x)$ converging to 0, whence $f^*(x) = 0$, that is, f^* vanishes on the real axis and is

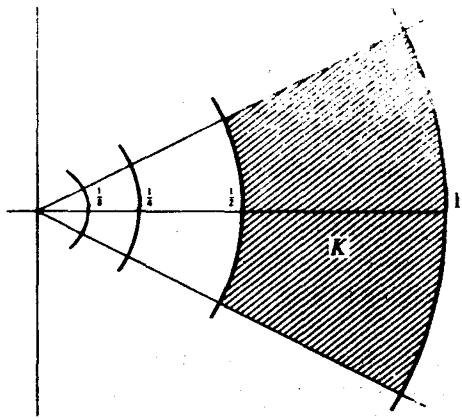

Fig. 1.

therefore identically 0. Since this argument holds for any convergent subsequence of $f_n(z)$, it follows that the original sequence $f_n(z)$ converged uniformly on K to 0. Therefore, for sufficiently large n, $|f_n(z)| < \varepsilon$ on K, which means $|f(z)| < \varepsilon$ on the set $|z| < 2^{-n}$. Thus $f(z)$ converges to 0 uniformly in the angle.

An extended real valued function $u(x)$ on a metric space X is *lower semicontinuous* if it never takes the value $-\infty$ (although $+\infty$ is permitted) and for every real λ the set defined by the inequality $u(x) \leqq \lambda$ is closed. The *upper semicontinuous* functions are the negatives of the lower semicontinuous ones, and the continuous functions are exactly those which are both upper and lower semicontinuous.

If K is a compact subset of X and $u(x)$ is lower semicontinuous on K, then that function is bounded from below on K, since otherwise the sets K_n

consisting of points x in K for which $u(x) \leqq -n$ would form a decreasing sequence of nonempty closed subsets of K; these would have to have a point in common at which the function took the excluded value $-\infty$. The function actually attains its infimum on K, for if $\lambda = \inf_{x \in K} u(x)$ then the subsets K_n of K defined by $u(x) \leqq \lambda + 1/n$ have a nonempty intersection K_λ upon which $u(x) = \lambda$.

Theorem: Let $f_\alpha(x)$ be a family of continuous (or lower semicontinuous) functions on the metric space X and $F(x) = \sup f_\alpha(x)$; then $F(x)$ is lower semicontinuous.

PROOF: It is evident that $F(x)$ cannot assume the value $-\infty$, and the set $F(x) \leqq \lambda$ is the intersection of the family of closed sets $f_\alpha(x) \leqq \lambda$ and is therefore closed.

A converse to the previous theorem holds if the space X is compact.

Theorem: Let $u(x)$ be lower semicontinuous on the compact metric space X; there then exists a monotone increasing sequence of continuous functions $u_k(x)$ converging to $u(x)$.

PROOF: Since the function $u(x)$ is bounded from below, there is no loss of generality in assuming that $u(x)$ is nonnegative on X. The compact metric space X is separable, and the open sets have a countable base, namely, the spheres $S(x_k, r)$ of rational radius centered about points of a given countable dense subset of X. For every pair of such spheres, S' and S'' where S' is contained in S'', we select once and for all a continuous function $f(x, S', S'')$ taking values in the interval $[0, 1]$, which vanishes outside S'' and equals $+1$ on S'. Only a countable family of functions $f(x, S', S'')$ is obtained in this way. For a given positive ε and every point x_0 in X the set $u(x) > u(x_0) - \varepsilon$ is an open set containing x_0; there exists, therefore, a pair of spheres S' and S'' in the countable base such that x_0 is contained in S' which in turn is contained in S''. Let r be a rational number in the interval $(u(x_0) - 2\varepsilon, u(x_0) - \varepsilon)$; now $rf(x, S', S'')$ is a continuous function satisfying the inequality $rf(x, S', S'') \leqq u(x)$ everywhere on X. Only countably many functions of the form $rf(x, S', S'')$ appear, and it is obvious that $u(x)$ is the supremum of this family if ε approaches 0. If that family is enumerated in any way and written $g_m(x)$, the functions

$$u_k(x) = \max_{m < k} g_m(x)$$

form a monotone increasing sequence of continuous functions which converges to $u(x)$.

此为试读，需要完整PDF请访问：www.ertongbook.com

2. Infinite Products

Let a_n be a sequence of complex numbers; we consider the sequence of products: $p_n = \prod_{k=1}^{n} a_k = a_1 a_2 a_3 \cdots a_n$. Obviously, if one of the a_k is 0, the products p_n vanish for all large n and the sequence of products converges trivially to 0. We suppose, therefore, that none of the factors vanishes; it is then clear that the products p_n converge to a limit P which is not zero and which is finite if and only if $\log p_n$ converges to $\log P$ for an appropriate determination of the logarithm, and, therefore, if and only if the series $\sum \log a_k$ is convergent. Now, if that series does not converge absolutely, it will be possible, by a suitable rearrangement of its terms, to make it converge to some other limit or to diverge. Accordingly, the partial products converge to a finite, nonzero limit independently of the order of the factors, if and only if the series $\sum |\log a_k|$ converges to a finite sum.

From the convergence of this series we deduce $\lim_k \log a_k = 0$, and therefore $\lim_k a_k = 1$ as we would expect. In studying the convergence of the product, then, we can assume that the numbers a_k are sufficiently close to 1.

Consider that determination of the logarithm which is real on the real axis; we have $\log 1 = 0$ and the logarithm is analytic in a circle of radius 1 about $z = 1$. We can divide that function by $(z - 1)$ to obtain a quotient $\log z/(z - 1) = q(z)$ which is analytic in the same circle and such that $q(1) = 1$. It is now clear that there exists $R > 0$ such that in a circle about $z = 1$ of radius R, $\frac{1}{2} < |q(z)| < 2$, and therefore,

$$\frac{|z|}{2} < |\log(1 + z)| < 2|z|, \quad \text{for all} \quad |z| < R.$$

We write $a_k = 1 + b_k$ and know that b_k converges to 0. Hence, for large k,

$$\frac{|b_k|}{2} < |\log(1 + b_k)|$$

$$= |\log a_k| < 2|b_k|$$

and the convergence of the series of logarithms is completely equivalent to the convergence of the series $\sum |b_k|$.

To sum up: The infinite product $\prod_{k=1}^{\infty}(1 + b_k)$ converges to a finite, nonzero limit independently of the order of the factors, *if and only if* the series $\sum |b_k|$ converges and no b_k is -1.

As an example we consider the infinite product

$$\prod_1^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)=P(z).$$

For any fixed value of z, the series $\sum|z^2/n^2\pi^2|$ evidently converges, thus $P(z)$ is well defined and finite for all z, and vanishes for z of the form $n\pi$ and only at such z. If $P_m(z)$ is the mth partial product and $|z| \leqq R$, then

$$\begin{aligned}|P_m(z)| &= \left|\prod_1^m\left(1-\frac{z^2}{n^2\pi^2}\right)\right| \\ &\leqq \prod_1^m\left(1+\frac{R^2}{n^2\pi^2}\right) \\ &= P_m(iR) \\ &< \prod_1^{\infty}\left(1+\frac{R^2}{n^2\pi^2}\right) = P(iR).\end{aligned}$$

It follows that the sequence of partial products is uniformly bounded in the circle of radius R, and hence contains a subsequence converging uniformly on that circle to an analytic limit which necessarily has the value $P(z)$ at z. The uniqueness of the limit shows that the passage to a subsequence was unnecessary, and since R was arbitrary, it follows that $P(z)$ is an entire function. Although true, it is not so clear that $P(z) = \sin z/z$. A proof will be given in Section 44.

Let us recall another theorem from the theory of functions:

Schwarz's Lemma: Let $f(z)$ be analytic in the circle $|z| < 1$ and bounded there; set $M = \sup|f(z)|$, $|z| < 1$ and suppose $f(0) = 0$. Then the function $h(z) = f(z)/z$ is also analytic in the circle and $\sup|h(z)| = M$.

PROOF: From the power series expansion we see that we can divide out z, and so $h(z)$ is analytic in the circle. We pass to a subcircle of radius $r = 1 - \varepsilon$ where the positive ε is small. For that subcircle, the function $|h(z)|$ assumes its maximum on the boundary, and that maximum is therefore of the form

$$\begin{aligned}|h(re^{i\omega})| &= \left|\frac{f(re^{i\omega})}{r}\right| \\ &\leqq \frac{M}{1-\varepsilon}.\end{aligned}$$

Since ε is arbitrarily small, $|h(z)|$ is bounded by M.

We obtain virtually the same result if we change the hypothesis slightly and suppose that $f(a) = 0$ for some a in the unit circle and divide out the function

$$h_a(z) = \frac{|a|}{a} \frac{a - z}{1 - \bar{a}z};$$

this linear fractional function has a zero at a, a pole at $1/\bar{a}$ and is regular in a neighborhood of $|z| = 1$ where it has absolute value 1. (Check this by computing the absolute value of $h(e^{i\omega})$.) If we divide $f(z)$ by $h_a(z)$, we find, as before, that the quotient has the same bound M in the circle.

We consider next a function $f(z)$, analytic in the circle and bounded by M; we suppose also that $f(0) = p > 0$. Let a_k be the sequence of zeros of $f(z)$; in general, this sequence is infinite, and we find it convenient to enumerate the zeros in such a way that

$$0 < |a_1| \leqq |a_2| \leqq |a_3| \leqq \cdots.$$

It should be noted that if a is a zero of order ν, then a occurs ν times in the sequence. Thus, each zero is counted as often as its multiplicity requires. Then, successively, for each a_k, we divide out $h_{a_k}(z)$, the quotient each time being bounded by M. In particular, at the origin we have, for any n,

$$0 < \frac{p}{\prod_1^n |a_k|} \leq M, \qquad \text{whence} \qquad \frac{p}{M} \leqq \prod_{k=1}^{n} |a_k|.$$

Since $|a_k|$ is always positive and smaller than 1, the sequence of partial products diminishes to a nonzero limit and hence the infinite product converges. We deduce that the series $\sum (1 - |a_k|)$ converges. Thus we have proved half of the following theorem, due to W. Blaschke.

Theorem (Blaschke): A sequence a_k of complex numbers in the unit circle is the set of zeros of a bounded analytic function with appropriate multiplicity if and only if the series $\sum (1 - |a_k|)$ converges.

Note that $1 - |a|$ is the distance from a to the boundary of the circle. To complete the proof of the Blaschke theorem, we construct, for a given sequence a_k satisfying the condition, a bounded analytic function having exactly those zeros. We may suppose that no a_k is zero. The function in question is the Blaschke product

$$B(z) = \prod_{k=1}^{\infty} h_{a_k}(z).$$

which vanishes whenever $z = a_k$. For other values of z, the product converges to a finite, nonzero limit, since

$$-b_k(z) = 1 - h_{a_k}(z)$$
$$= (1 - |a_k|)\frac{a_k + z|a_k|}{a_k(1 - \bar{a}_k z)},$$

whence $|b_k(z)| \leqq 2(1 - |a_k|)/(1 - |z|)$ and therefore $\sum |b_k(z)|$ converges. The partial products $B_n(z) = \prod_{k=1}^{n} h_{a_k}(z)$ are rational functions which have the bound 1 in the unit circle; the sequence of these products is then uniformly bounded for $|z| < 1$. Hence, there exists a subsequence converging to a limit $B(z)$ which is analytic in the circle; $B(z)$ must then coincide with the infinite product. This completes the proof. Note that by an ingenious choice of the numbers a_k we can construct a bounded analytic $B(z)$ which has the circle $|z| = 1$ as a natural boundary.

The theorem of Blaschke and Schwarz's lemma combined permit us to set up a canonical factorization for functions bounded and analytic in the circle:

$$f(z) = Cz^m B(z) e^{g(z)},$$

where m is an integer $\geqq 0$, C a constant, $B(z)$ a Blaschke product, and $g(z) = u(z) + iv(z)$ is analytic with $u(z) \leqq 0$. The integer m is the multiplicity of the zero of $f(z)$ at the origin, if there is one, m being equal to 0 otherwise, and the Blaschke product is completely determined by the other zeros of $f(z)$. Because of the argument above, the ratio $h(z) = f(z)/z^m B(z)$ is bounded in the circle and C should be taken as its bound. It follows that the function $h(z)/C$ has no zeros in the circle and is bounded there by 1; its logarithm is therefore analytic in the circle with a negative real part.

3. Convex Functions

We shall consider only functions $f(x)$, real and finite, defined on an open interval (a, b).

Such a function is *midpoint convex* if and only if for all x, y in (a, b).

$$f\left(\frac{x + y}{2}\right) \leqq \frac{f(x) + f(y)}{2},$$

and is said to be *convex* if and only if for all x, y in (a, b) and all t in the closed interval $[0, 1]$

$$f(tx + (1 - t)y) \leqq tf(x) + (1 - t)f(y).$$

The convex functions are clearly midpoint convex: we have only to set $t = \frac{1}{2}$; we shall show that any midpoint convex function which satisfies reasonable further conditions is convex.

That there exist midpoint convex functions which are not convex is shown by the following example: consider the real numbers as a vector space over the field of rational numbers, and let $\{x_\lambda\}$ be a Hamel base; every x in R is representable in a unique way as a finite sum with rational coefficients

$$x = \sum c_\lambda(x) x_\lambda .$$

The coefficients $c_\lambda(x)$ are "linear" functions of x taking rational values; since $\frac{1}{2}$ is rational,

$$c_\lambda\left(\frac{x+y}{2}\right) = \frac{c_\lambda(x) + c_\lambda(y)}{2}$$

and, therefore, $c_\lambda(x)$ is midpoint convex. Since it is not the constant function, and assumes only rational values, $c_\lambda(x)$ is not continuous, and therefore not convex, since, as we shall see, convex functions are continuous.

Theorem: If $f(x)$ is midpoint convex and continuous, then $f(x)$ is convex.

PROOF: We first show, by induction on n, that the convexity inequality above holds for all x and y in (a, b) and all t of the form $p/2^n$. The inequality being shown for n, we pass to $n + 1$: let

$$\begin{aligned} z &= \frac{p}{2^{n+1}} x + \frac{q}{2^{n+1}} y \\ &= \frac{1}{2}\left[\frac{p}{2^n} x + \frac{r}{2^n} y + y\right], \quad \text{where } p + q = 2^{n+1}, \end{aligned}$$

and where we may suppose that $p < q$, whence $p < 2^n < q = 2^n + r$. Now

$$\begin{aligned} f(z) &\leqq \frac{1}{2}\left[f\left(\frac{px + ry}{2^n}\right) + f(y)\right] \\ &\leqq \frac{p}{2^{n+1}} f(x) + \frac{q}{2^{n+1}} f(y). \end{aligned}$$

Since the set of t of the form $p/2^n$ is dense in the unit interval, from the continuity of $f(x)$, we obtain the full convexity inequality, that is, $f(x)$ is convex.

Theorem: If $f(x)$ is midpoint convex and is discontinuous at a point x_0 in (a, b), then $f(x)$ is unbounded on every subinterval of (a, b), and hence everywhere discontinuous.

PROOF: We may suppose that the interval is of the form $(-a, a)$, that $x_0 = 0$ and that $f(0) = 0$. There exists a sequence x_n converging to 0 for which $f(x_n)$ converges to a limit $m \neq 0$; we may suppose $m > 0$, since, otherwise, we pass to $y_n = -x_n$ and use that sequence instead. Now the sequence $2x_n$ also converges to 0, and we have

$$\begin{aligned} 2f(x_n) &\leqq f(0) + f(2x_n) \\ &= f(2x_n) \end{aligned}$$

and therefore $\liminf f(2x_n) \geqq 2m$. Repeating the argument $\liminf f(4x_n) \geqq 4m$ and inductively $\liminf f(2^k x_n) \geqq 2^k m$. Thus $f(x)$ is not bounded near $x = 0$, and there even exists a sequence x_n converging to 0 upon which f converges to infinity.

Let z be an arbitrary point of the interval; the sequence $z + 2x_n$ converges to z, while

$$\begin{aligned} f(x_n) &= f\left(\frac{z + 2x_n - z}{2}\right) \\ &\leqq \frac{f(z + 2x_n) + f(-z)}{2}. \end{aligned}$$

Since the left-hand side converges to infinity with increasing n, the right-hand side also converges to infinity, whence $f(z + 2x_n)$ converges to infinity, and f is not bounded near the point z. Since z was arbitrary, it follows that $f(x)$ is bounded in the neighborhood of no point.

Since convex functions are bounded on subintervals, it follows that convex functions are continuous.

The following beautiful theorem is due to Sierpinski.

Theorem (Sierpinski): If $f(x)$ is midpoint convex and Lebesgue measurable, then it is convex.

PROOF: The theorem is a consequence of the following even stronger result proved by Ostrowski.

Theorem (Ostrowski): If $f(x)$ is midpoint convex and bounded on a set E which is Lebesgue measurable with positive measure, then $f(x)$ is convex.

此为试读,需要完整PDF请访问: www.ertongbook.com