

A BOOK OF PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

**M. Krasnov, Cand.Sc., A. Kiselev, Cand.Sc.
and G. Makarenko, Cand.Sc.**

This problem book contains exercises for courses in differential equations at technical institutes. Particular attention is given to the method of isoclines for equations of the first and second order, to problems of finding orthogonal trajectories, to the use of the method of superposition in solving linear differential equations of order n , to linear dependence and linear independence of a system of functions, to problems of solving linear equations with constant and variable coefficients, to boundary value problems for differential equations, to integrating equations in power series, to asymptotic integration, to integrating systems of differential equations, to Lyapunov stability, and to the operator method.

Each section begins with a summary of basic facts and comprises worked-out examples of typical problems.

The book contains 967 problems a considerable portion of which have been composed by the authors themselves. There are some exercises of a theoretical nature.

The book is intended for students of technical institutes.

A BRIEF COURSE OF HIGHER MATHEMATICS

**V. Kudryavtsev, D.Sc. and
B. Demidovich, D.Sc.**

This textbook was written as a course in higher mathematics for university students majoring in various fields of natural sciences.

The text consists of twenty six chapters and two appendices. Each chapter contains a large number of test questions and exercises to make the subject matter more readily comprehensible. Answers are provided for all problems in the main text, and every effort has been made to ensure that they are accurate.

Translated from the fifth (1978) Russian edition.

VECTOR ANALYSIS

**M. Krasnov, Cand.Sc., A. Kiselev, Cand.Sc.
and G. Makarenko, Cand.Sc.**

This text is designed for students of engineering colleges and also for practicing engineers who feel a need to refresh their knowledge of such an important area of higher mathematics as vector analysis. Each section of the text starts out with a brief review of the essentials of theory (propositions, definitions, formulas). This is followed by detailed solutions of examples and problems.

The book contains about 300 problems to be solved by the student. They are all provided with hints and answers. The 41 drawings that accompany the text serve to help the reader in analyzing the theoretical material and problems.

V.A. ILYIN and E.G. POZNYAK

Fundamentals of Mathematical Analysis

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by Irene Aleksanova*

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В. А. ИЛЬИН, Э. Г. ПОЗНЯК

ОСНОВЫ
МАТЕМАТИЧЕСКОГО АНАЛИЗА

Часть I

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PREFACE TO THE RUSSIAN EDITION

The book is based on the lectures delivered by the authors at the Physics Department of Moscow University.

In the course of systematic exposition of the material, the authors gave prominence to the most important concepts and theorems, the most significant of the latter being called fundamental theorems. The authors tried to give the formulations of new concepts and theorems not long before their direct application.

The sequence of presentation of the material corresponds to that accepted at the Physics Department of Moscow University. The chapter "Preliminary Information on the Main Concepts of Mathematical Analysis", in particular, comes before the exposition of the systematic course because this chapter deals with some very important physical problems and discusses mathematical means for their solution. Thus it becomes clear from the very beginning what problems and concepts constitute the subject matter of mathematical analysis. Our lecturing experience shows that such a preliminary elucidation of the range of problems treated in the course of the analysis makes it appreciably easier for the students to understand the basic mathematical concepts.

The authors paid due attention to the ever increasing role of computations and methods of approximation; that is why they tried to use algorithmic form in proving theorems and carrying out calculations wherever it was possible. In Chap. 12, in particular, the main emphasis is laid on the algorithmic aspect of the methods of approximation and only then is the method substantiated.

In addition to the main part of the material, the authors consider it possible to include some sections given in small type.

In writing the book, the authors borrowed some methodical techniques from the course of lectures by N.V. Efimov as well as from the well-known books by E. Goursat, De La Vallée Poussin and F. Franklin.

The authors express their deep gratitude to A.N. Tikhonov and A.G. Sveshnikov for their valuable ideas and instructions and also for their enormous help at all stages of writing this book.

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The authors consider it necessary to point out that to a large extent their pedagogical views were influenced by their talks with I.M. Vinogradov and A.A. Samarsky, to whom they are also very thankful.

V. Ilyin, E. Poznyak

PREFACE TO THE ENGLISH EDITION

The present book is a translation into English of the 4th Russian edition of *Fundamentals of Mathematical Analysis*, Part 1, published by Nauka in 1982.

The previous edition has been revised and expanded: some new material has been introduced and a number of changes made, which were called forth by the desire of the authors to improve the exposition and to reflect the growing role of computational techniques and algorithms.

The authors consider it their pleasant duty to express their gratitude to Irene Aleksanova whose thorough work on the translation of the book helped to eliminate a number of inaccuracies which occurred in the Russian edition.

The authors.

CONTENTS

Preface to the Russian Edition	5
Preface to the English Edition	6
Chapter 1. Preliminary Information on the Main Concepts of Mathematical Analysis	17
1.1. Mathematical Concepts Needed for the Description of Motion	17
1.2. Instantaneous Velocity and the New Mathematical Concepts Related to It	19
1.3. The Problem of Reconstructing the Law of Motion from the Velocity and the Related Mathematical Problems	27
1.4. Some Questions Arising in Solving the Problems on Computation of a Path	29
1.5. Concluding Remarks	33
Chapter 2. Theory of Real Numbers	35
2.1. Real Numbers	35
2.1.1. The properties of rational numbers	35
2.1.2. Measuring the segments on the number axis	37
2.1.3. Real numbers and the rule of their comparison	40
2.1.4. Approximation of real numbers by rational numbers	42
2.1.5. Sets of real numbers bounded above and below	43
2.2. Arithmetic Operations on Real Numbers. Basic Properties of Real Numbers	47
2.2.1. Definition of the sum of real numbers	47
2.2.2. Definition of the product of real numbers	49
2.2.3. Properties of real numbers	49
2.2.4. Some frequently employed relations	52
2.3. Some Concrete Sets of Real Numbers	52
Supplement 1 to Chapter 2. Decimal-to-Binary Conversion and Binary-to-Decimal Conversion of Numbers	53
1. Decimal-to-binary conversion of numbers	53
2. Binary-to-decimal conversion of numbers	55
Supplement 2 to Chapter 2. On the Errors in Rounding Off the Numbers in the Number Systems with Even and Odd Bases	55
Chapter 3. Limit of a Sequence	57
3.1. Number Sequences	57
3.1.1. Number sequences and operations on them	57
3.1.2. Bounded and unbounded sequences	58
3.1.3. Infinitely large and infinitesimal sequences	59
3.1.4. Basic properties of infinitesimal sequences	60
3.2. Convergent Sequences and Their Basic Properties	63
3.2.1. The concept of a convergent sequence	63
3.2.2. Basic properties of convergent sequences	64

3.2.3. Limiting process in inequalities	67
3.3. Monotonic Sequences	68
3.3.1. Definition of monotonic sequences	68
3.3.2. Test for convergence of a monotonic sequence	69
3.3.3. Some examples of convergent monotonic sequences	70
3.3.4. Number e	73
3.4. Some Properties of Arbitrary Sequences and Number Sets	74
3.4.1. Subsequences of number sequences	74
3.4.2. Limit points of a sequence	75
3.4.3. Existence of a limit point of a bounded sequence	77
3.4.4. Separation of a convergent subsequence	80
3.4.5. A necessary and sufficient condition of convergence of a sequence	82
3.4.6. Some properties of arbitrary number sets	84
Supplement 1 to Chapter 3. Stolz' Theorem	86
Supplement 2 to Chapter 3. On the Rate of Convergence of a Sequence Approaching $\sqrt[n]{a}$	90
 Chapter 4. The Concept of a Function. The Limiting Value of a Function. Continuity	 93
4.1. The Concept of a Function	93
4.1.1. A variable quantity and a function	93
4.1.2. Methods of specification of a function	95
4.2. The Concept of the Limiting Value of a Function	96
4.2.1. Definition of the limiting value of a function	96
4.2.2. Arithmetic operations on functions possessing limiting values	99
4.2.3. Comparing infinitesimal and infinitely large functions	100
4.3. The Concept of Continuity of a Function	103
4.3.1. Definition of continuity of a function	103
4.3.2. Arithmetic operations on continuous functions	105
4.3.3. Composite function and its continuity	105
4.4. Some Properties of Monotonic Functions	106
4.4.1. Definition and examples of monotonic functions	106
4.4.2. The concept of an inverse function. Monotonic function having an inverse	106
4.5. Basic Elementary Functions	110
4.5.1. Rational powers of positive numbers	110
4.5.2. Exponential function	112
4.5.3. Logarithmic function	116
4.5.4. Hyperbolic functions	117
4.5.5. Power function with any real exponent α	118
4.5.6. Trigonometric functions	121
4.5.7. Inverse trigonometric functions	124
4.6. Limits of Some Functions	127
4.6.1. Preliminary remarks	127
4.6.2. The limit of the function $\frac{\sin x}{x}$ at point $x = 0$ (the first remarkable limit)	127
4.6.3. The limit of the function $\left(1 + \frac{1}{x}\right)^x$ for $x \rightarrow \infty$ (the second remarkable limit)	128
4.7. Continuity and Limiting Values of Some Composite Functions	132

4.7.1. Continuity and limiting values of some composite functions	132
4.7.2. The concept of an elementary function. The class of elementary functions	136
4.8. Classification of Points of Discontinuity of a Function	137
4.8.1. Points of discontinuity of a function and their classification	137
4.8.2. Piecewise-continuous functions	140
Supplement to Chapter 4. Proof of the Assertion Made in 4.5.6	140
1. The proof of the uniqueness	140
2. The proof of the existence	143
Chapter 5. Fundamentals of Differential Calculus	150
5.1. A derivative. Its Physical and Geometrical Interpretations	150
5.1.1. Increment of an argument and of a function. Difference form of the condition of continuity	150
5.1.2. Definition of a derivative	151
5.1.3. Physical interpretation of a derivative	152
5.1.4. Geometrical interpretation of a derivative	152
5.1.5. Right-hand and left-hand derivatives	154
5.1.6. The concept of a derivative of a vector function	154
5.2. The Concept of a Differentiable Function	156
5.2.1. Differentiation of a function at a given point	156
5.2.2. The connection between the concepts of differentiability and continuity of a function	157
5.2.3. The concept of a differential of a function	157
5.3. Rules for Differentiating a Sum, a Difference, a Product and a Quotient	159
5.4. Calculating the Derivatives of a Power Function, Trigonometric Functions and a Logarithmic Function	162
5.4.1. The derivative of a power function with an integral-valued exponent	162
5.4.2. The derivative of the function $y = \sin x$	162
5.4.3. The derivative of the function $y = \cos x$	163
5.4.4. The derivatives of the functions $y = \tan x$ and $y = \cot x$	164
5.4.5. The derivatives of the function $y = \log_a x$ ($0 < a \neq 1$)	164
5.5. Theorem on the Derivative of an Inverse Function	165
5.6. Computing the Derivatives of an Exponential Function and of Inverse Trigonometric Functions	166
5.6.1. The derivative of the exponential function $y = a^x$ ($0 < a \neq 1$)	166
5.6.2. The derivatives of inverse trigonometric functions	167
5.7. The Rule of Differentiating a Composite Function	169
5.8. The Logarithmic Derivative. The Derivative of a Power Function with Any Real Exponent. The Table of the Derivatives of Basic Elementary Functions	171
5.8.1. The concept of the logarithmic derivative of a function	171
5.8.2. The derivative of a power function with any real exponent	171
5.8.3. The table of derivatives of basic elementary functions	172
5.9. Invariance of the Form of the First Differential. Some Applications of a Differential	173
5.9.1. Invariance of the form of the first differential	173
5.9.2. Formulas and rules for computing differentials	175
5.9.3. Application of differentials for establishing approximation formulas	175

5.10.	Derivatives and Differentials of Higher Orders	177
5.10.1.	The concept of a derivative of the n th order	177
5.10.2.	Derivatives of the n th order of some functions	177
5.10.3.	Leibniz' formula for the n th order derivative of the product of two functions	179
5.10.4.	Higher-order differentials	180
5.11.	Differentiation of a Function Represented Parametrically	182
Chapter 6.	Indefinite Integral	184
6.1.	The Concepts of an Antiderivative of a Function and of the Indefinite Integral	184
6.1.1.	The concept of an antiderivative of a function	184
6.1.2.	Indefinite integral	185
6.1.3.	Basic properties of the indefinite integral	186
6.1.4.	Table of the basic indefinite integrals	187
6.2.	Main Methods of Integration	190
6.2.1.	Integration by a change of variable (substitution)	190
6.2.2.	Integration by parts	193
Chapter 7.	Complex Numbers. Algebra of Polynomials Integration in Terms of Elementary Functions	197
7.1.	Brief Information on Complex Numbers	197
7.2.	Algebraic Polynomials	201
7.3.	Multiple Roots of Polynomials. Test for Distinguishing the Multiplicity of a Root	203
7.4.	The Principle of Separation of Multiple Roots. Euclidean Algorithm	205
7.4.1.	The principle of separation of multiple roots	205
7.4.2.	Finding the greatest common divisor of two polynomials (Euclidean algorithm)	206
7.5.	Decomposition of a Proper Rational Fraction with Complex Coefficients into a Sum of Partial Fractions	209
7.6.	Factorization of an Algebraic Polynomial with Real Coefficients into a Product of Irreducible Real Factors	211
7.7.	Decomposition of a Proper Rational Fraction with Real Coefficients into a Sum of Partial Fractions with Real Coefficients	213
7.8.	The Problem of Integrating Rational Fractions	218
7.9.	Ostrogradsky's Method	221
7.10.	Integrating Some Irrational and Transcendental Expressions	224
7.10.1.	Integrating certain trigonometric expressions	224
7.10.2.	Integrating linear-fractional irrational functions	227
7.10.3.	Integrating binomial differentials	228
7.10.4.	Integrating quadratic irrational functions by means of Euler's substitution	230
7.10.5.	Integrating quadratic irrational functions by other means	232
7.11.	Elliptic Integrals	238
Chapter 8.	Fundamental Theorems on Continuous and Differentiable Functions	241
8.1.	A New Definition of the Limiting Value of a Function	241
8.1.1.	A new definition of the limiting value of a function. Its equivalence to the definition given earlier	241

8.1.2. The necessary and sufficient condition of the existence of a limiting value (Cauchy's test)	244
8.2. A Local Boundedness of a Function Having a Limiting Value	246
8.3. Theorem on the Stability of Sign of a Continuous Function	248
8.4. The Passage by a Continuous Function Through Any Intermediate Value	249
8.4.1. The passage by a continuous function through zero at a change of signs	249
8.4.2. The passage by a continuous function through any intermediate value	250
8.5. The Boundedness of a Function Continuous on a Closed Interval	250
8.6. The Least Upper Bound and the Greatest Lower Bound of a Function and Their Attainment by a Function Continuous on a Closed Interval	251
8.6.1. The concepts of the least upper bound (the supremum) and the greatest lower bound (the infimum) of a function on a given set	251
8.6.2. Attaining by a function, continuous on a closed interval, its least upper and greatest lower bounds	252
8.7. A Function Increasing (Decreasing) at a Point. A Local Maximum (Minimum)	254
8.7.1. An increase (decrease) of a function at a point	254
8.7.2. A local maximum and a local minimum of a function	255
8.8. Theorem on a Zero of a Derivative	256
8.9. The Formula of Finite Increments (Lagrange's Formula)	257
8.10. Some Corollaries of Lagrange's Formula	258
8.10.1. The constant value of the function possessing a zero derivative on an interval	258
8.10.2. Conditions for monotonicity of a function on an interval	259
8.10.3. A derivative possessing no points of discontinuity of the 1st kind or removable discontinuity	261
8.10.4. Derivation of certain inequalities	262
8.11. A Generalized Formula of Finite Increments (Cauchy's Formula)	263
8.12. Evaluation of Indeterminate Forms (L'Hospital's Rule)	264
8.12.1. Evaluating indeterminate expressions of the form $\frac{0}{0}$	264
8.12.2. Evaluating indeterminate expressions of the form $\frac{\infty}{\infty}$	266
8.12.3. Evaluating indeterminate expressions of other forms	268
8.13. Taylor's Formula	269
8.14. Various Forms of the Remainder, Maclaurin's Formula	272
8.14.1. The remainder in the form of Lagrange, Cauchy and Peano	272
8.14.2. Another form of Taylor's formula	275
8.14.3. Maclaurin's formula	275
8.15. Evaluation of the Remainder. Expansion of Some Elementary Functions	276
8.15.1. Evaluating the remainder for an arbitrary function	276
8.15.2. Expanding some elementary functions by Maclaurin's formula	277
8.16. Examples of Applications of Maclaurin's Formula	280
8.16.1. Algorithm for calculating the number e	280

8.16.2.	Realization of algorithm for calculating the number e by means of a computer	280
8.16.3.	Application of Maclaurin's formula for asymptotic estimations of elementary functions and for computing limits	282
	Supplement to Chapter 8. Calculation of Elementary Functions	285
1.	Calculation of the logarithmic function and the inverse trigonometric functions	285
2.	Computing the trigonometric functions, the exponential function and the hyperbolic functions	288
Chapter 9.	Geometric Investigation of the Graph of a Function. Finding the Maximum and Minimum Values of a Function	295
9.1.	Intervals of Monotonicity of a Function. Finding the Points of Extremum	295
9.1.1.	Determining the intervals of monotonicity of a function	295
9.1.2.	Finding the points of a possible extremum	295
9.1.3.	The first sufficient condition of extremum	296
9.1.4.	The second sufficient condition of extremum	298
9.1.5.	Extremum of a function nondifferentiable at a given point. A general scheme of searching for extrema	301
9.2.	The Direction of the Convexity of the Graph of a Function	304
9.3.	Point of Inflection of the Graph of a Function	306
9.3.1.	Definition of the point of inflection. A necessary condition for inflection	306
9.3.2.	The first sufficient condition of inflection	309
9.3.3.	The second sufficient condition of inflection	309
9.3.4.	Some generalizations of the first sufficient condition of inflection	310
9.4.	The Third Sufficient Condition of an Extremum and an Inflection	311
9.5.	Asymptotes of the Graph of a Function	317
9.6.	The Scheme for Investigating the Graph of a Function	316
9.7.	Finding the Maximum and the Minimum Value of a Function. An End-Point Extremum	319
9.7.1.	Finding the maximum and minimum value of a function	319
9.7.2.	End-point extremum	322
Chapter 10.	Definite Integral	324
10.1.	Integral Sums. Integrability	324
10.2.	Upper and Lower Sums	327
10.2.1.	The concept of an upper and a lower sum	327
10.2.2.	The properties of the upper and lower sums	328
10.3.	A Necessary and Sufficient Condition of Integrability	332
10.4.	Some Classes of Integrable Functions	334
10.4.1.	The property of uniform continuity of a function	334
10.4.2.	Heine-Borel lemma. Another proof of the theorem on uniform continuity	337
10.4.3.	Integrability of continuous functions	338
10.4.4.	Integrability of some discontinuous functions	339
10.4.5.	Integrability of monotonic bounded functions	341
10.5.	The Main Properties of the Definite Integral	341
10.6.	Evaluations of Integrals. Mean-Value Formulas	344
10.6.1.	Evaluations of integrals	344
10.6.2.	The first mean-value formula	347

10.6.3. The first mean-value formula in a generalized form	348
10.6.4. The second mean-value formula	349
10.7. The Existence of an Antiderivative of a Continuous Function. Basic Rules of Integration	349
10.7.1. The existence of an antiderivative of a continuous function	349
10.7.2. The fundamental formula of integral calculus	351
10.7.3. A change of variable under the definite integral sign	353
10.7.4. The formula of integration by parts	354
10.7.5. The remainder in Taylor's formula in integral form	355
Supplement 1 to Chapter 10. Some Important Inequalities for Sums and Integrals	357
1.1. Derivation of a preliminary inequality	357
1.2. Hölder's inequality for sums	358
1.3. Minkowski's inequality for sums	359
1.4. Integrability of an arbitrary positive degree of the modulus of the function being integrated	359
1.5. Hölder's inequality for integrals	360
1.6. Minkowski's inequality for integrals	362
Supplement 2 to Chapter 10. Proof of the Assertion from 10.6.4	362
Chapter 11. Applications of the Definite Integral in Geometry and Physics	365
11.1. The Length of the Arc of a Curve	365
11.1.1. The concept of a plane curve	365
11.1.2. Parametric representation of a curve	366
11.1.3. The notion of a space curve	368
11.1.4. The concept of an arc length of a curve	369
11.1.5. Sufficient conditions for rectifiability of a curve. Formulas for computing the arc length of a curve	373
11.1.6. Differential of an arc	378
11.1.7. Some examples of arc length computation	379
11.2. Area of a Plane Figure	380
11.2.1. The concept of squareability of a plane figure. Area of a squareable plane figure	380
11.2.2. Area of a curvilinear trapezoid	382
11.2.3. Area of a curvilinear sector	383
11.2.4. Examples of computing areas	385
11.3. Volumes of Bodies and Areas of Surfaces	386
11.3.1. The concept of cubability and volume	386
11.3.2. Cubability of some classes of bodies	387
11.3.3. Examples of computing volumes	389
11.3.4. Area of a surface of revolution	389
11.4. Some Physical Applications of the Definite Integral	392
11.4.1. The mass and the centre of gravity of a nonhomogeneous bar	392
11.4.2. The work performed by a variable force	393
Supplement to Chapter 11. An Example of a Nonsquareable Figure	394
Chapter 12. Approximate Methods of Calculating Roots of Equations and Definite Integrals	399
12.1. Approximate Methods of Calculating Roots of Equations	399
12.1.1. The "fork" method	399
12.1.2. The tangent method	400

12.1.3. The chord method	401
12.1.4. The method of iterations (successive approximations)	402
12.1.5. Substantiation of the tangent method	405
12.1.6. Substantiation of the chord method	409
12.2. Approximate Methods of Calculating Definite Integrals	411
12.2.1. Introductory remarks	411
12.2.2. The method of rectangles	414
12.2.3. The method of trapezoids	417
12.2.4. The method of parabolas	419
12.2.5. Concluding remarks	422
Chapter 13. Theory of Number Series	424
13.1. The Concept of a Number Series	424
13.1.1. A series and its partial sums. Convergent and divergent series	424
13.1.2. Cauchy's test for convergence of a series	427
13.1.3. Two properties connected with the convergence of a series	429
13.2. Series with Positive Terms	429
13.2.1. A necessary and sufficient condition for convergence of a series with positive terms	429
13.2.2. Comparison tests	430
13.2.3. D'Alembert's and Cauchy's tests	433
13.2.4. Cauchy-Maclaurin's integral test	437
13.2.5. Raabe's test	440
13.2.6. Absence of a universal series of comparison	442
13.3. Absolutely and Conditionally Convergent Series	443
13.3.1. The concept of an absolutely and conditionally convergent series	443
13.3.2. On rearrangement of terms of conditionally convergent series	445
13.3.3. On rearrangement of an absolutely convergent series	448
13.4. Arithmetic Operations on Convergent Series	450
13.5. Tests of Convergence of Arbitrary Series	452
13.5.1. Leibniz's test	452
13.5.2. The test of Dirichlet-Abel	454
13.6. Infinite Products	457
13.6.1. Basic notions	457
13.6.2. Connection between the convergence of continued products and series	459
Supplement 1 to Chapter 13. Auxiliary Theorem to 13.2.3	463
Supplement 2 to Chapter 13. Expansion of the Function $\sin x$ into a Continued Product	464
Supplement 3 to Chapter 13. Generalized Methods of Summation of Divergent Series	468
Chapter 14. Functions of Several Variables	472
14.1. The Concept of a Function of Several Variables	472
14.1.1. On functional relationships between several variable quantities	472
14.1.2. The concepts of the Euclidean plane and the Euclidean space	472
14.1.3. The concept of a function of two or three variables	473
14.1.4. The concepts of the m -dimensional coordinate space and m -dimensional Euclidean space	475
14.1.5. The sets of points of the m -dimensional Euclidean space E^m	476