STUDIES IN APPLIED MECHANICS 16

Developments in Engineering Mechanics

Edited by

A.P.S. Selvadurai

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Edited by

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PREFACE

The year 1987 marks the 100th Anniversary of the formation of the first Canadian Learned Engineering Society, the Canadian Society of Civil Engineering, which was renamed the Engineering Institute of Canada in 1918. The Centennial is therefore of special significance to the Canadian community of Civil Engineers. The Canadian Society for Civil Engineering (CSCE) Centennial Conference forms a part of the Canadian Engineering Centennial Convention which was organized for this occasion.

The Technical Sessions on *Developments in Engineering Mechanics* form a part of the CSCE Centennial Conference and were held at the Palais de Congres in Montreal, Quebec, from 18th to 22nd May 1987. These Technical Sessions were sponsored by the newly formed Engineering Mechanics Division of the CSCE. The objectives of these Technical Sessions were to highlight the contributions made to the field of Engineering Mechanics by leading researchers in Canada. The sessions organized covered a diversity of topics, primarily in Solid Mechanics, including Dynamics and Stability of Flexible Structures, Non-linear Elasticity, Stability of Inflatable and Shell Structures, Composite Structures, Mechanics of Yield, Failure and Damage, Computer Modelling, Computer Applications, Geomechanics and Experimental Mechanics. The contributions recorded in this volume address the state-of-the-art and current developments in the various topics.

The Editor wishes to thank the authors for their willingness to contribute to the volume and for their participation at the Technical Sessions of the CSCE Centennial Conference. The Editor gratefully acknowledges the support and encouragement of Professor M.S. Mirza (McGill University), Past President of the CSCE whose efforts led to the organization of the Engineering Mechanics Division within the CSCE. Thanks are also due to Mrs. S.J. Selvadurai for editorial assistance and for preparation of the subject index.

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ON NON-SELFADJOINT PROBLEMS IN MECHANICS

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ABSTRACT

In this paper, stability of elastic structures is investigated. It is shown that an instability process is essentially a dynamic one. The static approach to stability can only be followed in the case the problem is selfadjoint. However, numerous problems, also in civil engineering, are non-selfadjoint. Therefore, essentially a dynamic approach to these problems must be used. It is shown how to handle such problems mathematically, and it is shown that they can specifically occur in the control of elastic structures.

INTRODUCTION

In dealing with stability problems, the civil engineer is used to consider these as static problems. Take as an example Euler's column, Fig. 1. There is a load P involved that is said to cause buckling when exceeding a certain limit of magnitude. Apparently, no dynamic effects occur. A well established theory for these so-called "static stability problems" exists. It was initiated by Euler (ref.1), developed further by G. H. Bryan (ref.2), E. Trefftz (ref.3), St. P. Timoshenko (ref.4), and many others. The civil engineer feels comfortable studying this theory and remaining, thus, completely in the domain of statics.

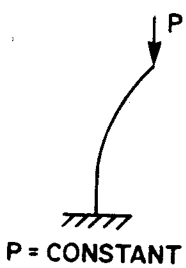

Fig. 1. Euler's column.

However, there exists also a subject denoted "dynamic stability". In that case, a time dependent, not static loading is present. A simple example is a column carrying a compressive load P^* that is not constant, like for Euler's column, but is a *harmonic function of time*, Fig. 2. Under such circumstances, one is of course not surprised that problems of this nature have to be treated by means of dynamics. A corresponding theory has been provided among others by V. V. Bolotin (ref.5). Yet, a civil engineer might consider the case of dynamic stability as an exception which, should it ever occur, had to be dealt with by an appropriate expert who would use a special theory.

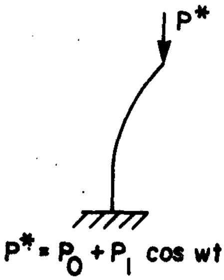

Fig. 2. Beliaev's column.

Recently, also a theory of "nonconservative stability" has been introduced involving so-called follower forces. It is claimed that, when for a column a *tangential* follower force P_t is present, Fig. 3, flutter instead of buckling takes place for a critical value of P_t although the follower force is not time dependent. Therefore, something puzzling happens: Flutter, a truly dynamic process, occurs in spite of all quantities involved in the mathematical description of the column being time independent. A civil engineer may consider such a situation, half incredulous, as something fancy he may not have to bother with, although a well established theory of nonconservative stability of elastic systems has been set up by V. V. Bolotin (ref.6), H. Ziegler (ref.7), and H. Leipholz (ref.8).

The truth is that any instability process, buckling or flutter, is actually a dynamic one. Even in the case of the buckling of an ordinary Euler column, a motion, and therefore a dynamic process takes place: According to Euler's theory the column passes in the course of buckling through various equilibrium positions that are adjacent to the original (trivial) one. That is not possible without a *motion* of the column. Hence, there is indeed a dynamic effect present which is simply suppressed though in the deliberations of "static" instability. Moreover, it can be shown that it is not the so-called

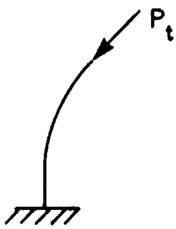

Fig. 3. Beck's column.

"buckling load" which makes a column buckle. This load makes the column only prone to buckle by acting as a destabilizing structural parameter. The actual buckling is caused by the influence of one of the ever present external "perturbations", for example an impact, an impulse loading, and these are certainly dynamic loadings. The reason why this is not so clear follows from the fact that, again, the occurrence of a perturbation is ignored in the theory of "static stability" like the *motion* of the column through a sequence of equilibrium positions during the buckling process had been ignored.

All this is also the case when instability is caused by a follower load. As before, this load has only the role of a destabilizing structural parameter while the instability is caused by a perturbation which is also not mentioned explicitely in the theory. In both cases, (Euler buckling and flutter owing to a critical value of the follower force), the actual instability is of a dynamic nature. The reason why that is not equally obvious in both cases is that flutter is violent and very noticeable while the moving of a column through a number of adjacent equilibrium positions is only a mild form of a dynamic process.

At this point, the conclusion can be drawn that a theory of dynamic stability should be adopted as an all embracing theory of a unifying character and that civil engineers should become acquainted with it in order to be capable to deal with modern stability problems of all kinds. Subsequently, it will be shown that for example in the context of *structural control*, which appears to be an emerging facet of modern civil engineering, specifically situations involving flutter can occur. Since flutter systems are non-selfadjoint, also the question how such problems can be handled mathematically is of importance and will be considered.

DESTABILIZING PARAMETER, DEGREE OF STABILITY, AND PERTURBATION

To fix the preceding ideas, let an analytical approach to an example be presented. This example is still so simple that the basic concepts will come through very clearly.

Consider the motion of a mechanical system after it has been subjected to an external perturbation. This motion is analytically described by

$$\left.\begin{aligned} &L[q(t),\ P,\ t] = f(t), \\ &q(0) = q_0,\quad \dot{q}(0) = \dot{q}_0, \end{aligned}\right\} \qquad \text{(a)}$$

where L is a differential operator, $q(t)$ the "characteristic" of the system whose stability is to be determined, P the destabilizing parameter, t the time, $f(t)$ the perturbation.

It is common to assume an initial impulse to be the perturbation. Then

$$f(t) = I\delta(t),\quad I = m\dot{q}_0,\quad \delta(t) = \text{Dirac function} \qquad \text{(b)}$$

Since $\delta(t) = 0$ except at $t = 0$, equation (a) can be replaced for $t > 0$ by

$$\left.\begin{aligned} &L[q(t),\ P,\ t] = 0, \\ &q(0) = q_0,\ \dot{q}(0) = \frac{I}{m}\ . \end{aligned}\right\} \qquad \text{(c)}$$

The advantage of working with impulses is that the stability problem as represented by (c) involves a homogeneous differential equation only, and the perturbation is expressed simply in terms of the initial conditions. Therefore it is justified to talk in this context of "perturbations of the initial conditions", an expression frequently used when stability is investigated.

Let the differential equation in (a) and the perturbation in (b) be written down for the buckling problem shown in Fig. 4. The potential energy of this system is

$$V = \frac{c}{2}\,\theta^2 - \frac{P\ell}{2}\,\theta^2. \qquad \text{(d)}$$

Its kinetic energy is

$$T = \frac{m}{2}\,a^2\dot{\theta}^2. \qquad \text{(e)}$$

Lagrange's technique yields as linearized equation of motion the equation

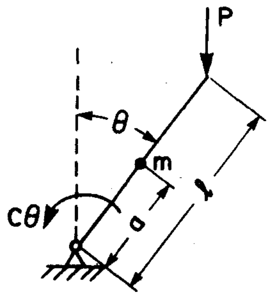

Fig. 4. Rigid column bearing a masspoint at midpoint and having a restoring moment at the lower end.

$$ma^2\ddot{\theta} - (P\ell - c)\theta = I\delta(t). \tag{f}$$

Comparing (f) with (a) shows that $q(t) \equiv \theta$ is the characteristic of the system, P is the destabilizing parameter, and $f(t) = I\delta(t)$ as assumed in (b). Moreover,

$$I = ma\dot{\theta}(0) \tag{g}$$

is the magnitude of the initial impulse that causes the system to vibrate about its trivial equilibrium position.

The solution to (f) for $\theta(0) = 0$, $\dot{\theta}(0)$ = const. is

$$\theta(t) = \frac{Ia}{\sqrt{m(c - P\ell)}} \int_0^t \sin\sqrt{\frac{c - P\ell}{ma^2}}\,(t - \xi)\delta(\xi)d\xi,$$
$$\theta(t) = \frac{Ia}{\sqrt{m(c - P\ell)}} \sin\sqrt{\frac{c - P\ell}{ma^2}}\,t. \tag{h}$$

For $c \neq P\ell$, it is a harmonic oscillation with the amplitude

$$A = \frac{Ia}{\sqrt{m(c - P\ell)}}\,. \tag{k}$$

A discussion of (k) shows that for $P < c/\ell = P_{crit}$, no impulse of finite magnitude I could cause the vibration (h) of the column to assume an infinitely large amplitude A. As long as $P < P_{crit}$, instability of the column, i.e. $A = \infty$, would be possible for an infinite I only. However, I represents the *degree of stability*. Hence, the column in Fig. 4 is a system with an infinite degree of stability in the context of a linear theory.

Let the value of the *destabilizing parameter* P approach P_{crit}. The impulse I in

$$I = \frac{A}{a} \sqrt{m(c - P\ell)}, \qquad (\ell)$$

decreases as P tends towards $P_{crit} = c/\ell$. It becomes zero for $P = P_{crit} = c/\ell$. At that point of time, the column becomes unstable for any arbitrarily small perturbing impulse.

This discussion shows that, as claimed before, load P is not the "buckling load". It is only the catalyst of instability but not its cause. The cause for instability is now as ever the perturbation, as small it might be for $P = P_{crit}$.

A detailed discussion of the concepts of "destabilizing parameter, degree of stability, and perturbation" can also be found in (ref.9).

SELFADJOINT STABILITY PROBLEMS

Stability problems in civil engineering are in reality not as simple as assumed in the preceding section. The simple presentation of the previous column buckling was only possible, because the column was interpreted as a lumped mass system. But civil engineering structures involve elements with distributed loadings, masses, and stiffnesses. Therefore, the mathematical description must be more sophisticated using partial differential equations.

Consider the small vibrations of a mechanical system about its trivial equilibrium position caused by an external perturbation. The mathematical formulation of this problem is given by

$$\mu\ddot{w}(x, t) + E[w(x, t)] + PC[w(x, t)] = F(x, t), \qquad (1)$$

$$\{U_i[w(x, t)]\}_B = 0, \quad i = 1,2,\ldots,2n, \qquad (2)$$

$$w(x, 0) = f(x), \quad \dot{w}(x, 0) = g(x). \qquad (3)$$

Eq. (1) represents a partial differential equation, eq. (2) stands for the boundary conditions, and eq. (3) for the initial conditions.

Furthermore, in (1), μ is the mass per unit dimension (length, surface or volume), w the deflection, x the "vector" of spatial coordinates, t the time, P the destabilizing parameter, E a linear differential operator of order 2n describing the elastic properties of the system, C a linear differential operator of maximum order 2n-2, and F the function modelling the external perturbation of the system. Finally, ($\ddot{\ }$) denotes double differentiation with respect to time.

In (2), the U_i are linear differential operators, and $\{\cdots\}_B$ indicates that the expression in the curled bracket is to be taken at certain parts of the boundary B of the system. In (3), $f(x)$ and $g(x)$ are two given functions that are sufficiently smooth and satisfy the boundary conditions, and ($\dot{\ }$) denotes differentiation with respect to time.

The nature of problem (1), (2) and (3) depends on the nature of operator C. If C is selfadjoint, the whole problem is *selfadjoint*, i.e. *symmetric*. If C is *non-selfadjoint*, so is the whole problem, and therefore *unsymmetric*.

In the selfadjoint case, eq. (1) assumes the form

$$\mu\ddot{w}(x,\ t) + E[w(x,\ t)] + PS[w(x,\ t)] = F(x,\ t), \tag{4}$$

where the operator C has been replaced by S which is supposed to be selfadjoint. Boundary and initial conditions (2), (3) remain unchanged.

Introduce the so-called *mode generating problem*

$$-\mu\omega_k^2 y_k(x) + E[y_k(x)] + PS[y_k(x)] = 0, \tag{5}$$

$$\{U_i[y_k(x)]\}_B = 0, \tag{6}$$

where the y_k, $k = 1,2,3,\ldots$, are eigenfunctions representing the modes to problem (1), (2), (3), and the ω_k are the corresponding eigenvalues. By virtue of the selfadjointness of (4), (5) the functions y_k and the eigenvalues ω_k are real valued. Moreover, the y_k are orthonormal so that

$$\int_S y_k y_j dx = \delta_{kj}, \tag{7}$$

where S is the domain of integration of the system, and δ_{kj} is the Kronecker symbol. Also, the y_k are uniformly bounded. Hence,

$$|y_k| < M, \tag{8}$$

where M is an appropriate constant. Finally, owing to well-known expansion theorems (ref.10), the y_k form a relative complete system. Consequently,

$$f(x) = \sum_i a_i y_i(x), \quad g(x) = \sum_i b_i y_i(x), \tag{9}$$

as all real valued, continuous functions that satisfy the boundary conditions (2), i.e. (6), are expandable in terms of the y_k. Using the orthonormality of the "coordinate functions" y_k, the coefficients a_i and b_i in (9) are obtained as

$$a_i = \int_S y_i(x)f(x)dx, \quad b_i = \int_S y_i(x)g(x)dx. \tag{10}$$

By means of Mercer's theorem it can also be shown that

$$\sum_i \frac{1}{\omega_i^2} = K < \infty. \tag{11}$$

As will be seen, all these conditions are sufficient to warrant applicability of the modal approach.

By means of Laplace transformation, eq. (4) can be changed into

$$\mu p^2\bar{w} + E[\bar{w}] + PS[\bar{w}] = \bar{F} \tag{12}$$

if one assumes the initial conditions to be homogeneous (i.e. $f = g \equiv 0$). In (12),

$$\bar{w}(x, p) = \int_0^\infty e^{-pt}w(x, t)dt, \quad \bar{F}(x, p) = \int_0^\infty e^{-pt}F(x, t)dt. \tag{13}$$

Expand $\bar{w}$ and $\bar{F}$ in terms of the coordinate functions y_k in order to obtain

$$\bar{w}(x, p) = \sum_k A_k(p)y_k(x), \tag{14}$$

$$\bar{F}(x, p) = \sum_k B_k(p)y_k(x). \tag{15}$$

The coefficients A_k in (14) are yet undetermined. The coefficients B_k in (15) can to the contrary be found as

$$B_k(p) = \int_S \bar{F}(x, p)y_k(x)dx \tag{16}$$

owing to (7).

Using (14) and (15) in (12) yields

$$\sum_k \mu p^2 A_k y_k + E[\sum_k A_k y_k] + PS[\sum_k A_k y_k] = \sum_k B_k y_k. \tag{17}$$

Because E and C are linear, (17) can be rewritten as

$$\sum_k A_k\{\mu p^2 y_k + E[y_k] + PS[y_k]\} = \sum_k B_k y_k. \tag{18}$$

From (5) follows

$$E[y_k] + PS[y_k] = \mu\omega_k^2 y_k. \tag{19}$$

Substituting (19) in (18) yields

$$\sum_k A_k[\mu(p^2 + \omega_k^2)y_k] = \sum_k B_k y_k. \tag{20}$$

Comparing coefficients of y_k leads to

$$A_k = \frac{1}{\mu}\frac{B_k}{p^2 + \omega_k^2}. \tag{21}$$

Hence, by virtue of (14) and (21),

$$\bar{w}(x, p) = \sum_k \frac{B_k(p)}{\mu(p^2 + \omega_k^2)} y_k(x). \tag{22}$$

Owing to (16), (22) changes into

$$\bar{w}(x,p) = \sum_k \frac{y_k(x)}{\mu(p^2 + \omega_k^2)} \int_S \bar{F}(\xi, p) y_k(\xi) d\xi. \tag{23}$$

Applying inverse Laplace transformation to (23) yields

$$w(x, t) = \sum_k \int_S \int_0^t \frac{y_k(x) y_k(\xi)}{\mu\omega_k} \sin \omega_k(t - \tau) F(\xi, \tau) d\tau d\xi. \tag{24}$$

Integration by parts with respect to τ replaces (24) by

$$w(x, t) = -\sum_k \int_S \int_0^t \frac{y_k(x) y_k(\xi)}{\mu\omega_k^2} \cos \omega_k(t - \tau) \frac{\partial F(\xi, \tau)}{\partial \tau} d\tau d\xi$$

$$+ \sum_k \int_S \frac{y_k(x) y_k(\xi)}{\mu\omega_k^2} F(\xi, t) d\xi - \sum_k \int_S \frac{y_k(x) y_k(\xi)}{\mu\omega_k^2} F(\xi, 0) \cos \omega_k t d\xi. \tag{25}$$

Yet, by virtue of (8) and (11),