STOCHASTIC PROCESSES AND THE WIENER INTEGRAL

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PREFACE

The present book is an introduction to stochastic processes, Brownian motion processes, Gaussian processes, Wiener measure and Wiener integrals. Below is a brief outline of its contents.

In Chapter 1, it is shown that a stochastic process on an arbitrary probability space induces a probability measure in the space of all real valued functions and that the process can then be represented by a process on the function space. This is done by means of the Kolmogorov extension theorem, a detailed proof of which is given here. Separability, measurability and continuity of a stochastic process are discussed in this chapter. There is also a study of infinite dimensional random vectors. Chapter 2 is a brief treatise of martingales. The main objects here are the martingale convergence theorem and the martingale closing theorem. Chapter 3 begins with an existence proof of additive processes (i.e., processes with independence increments). This is followed by a discussion of sample function properties of additive processes. The Brownian motion process is then characterized as an additive process with continuous sample functions. In Chapter 4 the existence of a Gaussian process having an arbitrarily given pair of real valued function and positive definite symmetric function as its mean and covariance function is proved. In Chapter 5, the stochastic integral of stepwise stochastic process with respect to a continuous Brownian motion process is defined as a random variable which is the Riemann-Stieltjes integrals of sample functions of the former process with respect to those of the latter. This definition is then extended to a wider class of stochastic processes satisfying certain measurability and integrability conditions. In Chapter 6, Hájek's proof of the Feldman-Hájek dichotomy that two Gaussian measures on a function space are either equivalent or singular is given. In Chapter 7 the Wiener measure is first constructed on the space of all real valued functions. Then it is shown that the subspace consisting of the continuous functions has an outermeasure equal to 1 and thus inherits a probability measure from the containing space of all real valued functions. Chapter 8 contains the Cameron-Martin translation theorem of Wiener integrals. The translation theorem in § 34 is a particular case of a more general theorem in Cameron and Martin [2]. I presented it here because of the relative brevity of its proof and because of its application.

The bibliography is not intended to be complete. Rather, it is a list of publications which I referred to or drew material from in writing this book. Gelfand and Yaglom [1], Kovalchik [1], Yaglom [1] and Shepp [1] contain longer lists of publications in Wiener integrals and the Feldman-Hájek dichotomy up to the times of their publication.

The prerequisite for reading this book is a good background in real analysis and some knowledge of measure theoretic probability theory. In the Appendix I collected the theorems in probability theory along with definitions needed in stating them. The numberings of these definitions and theorems are preceded by the letter A. Thus for instance Theorem A4.2 appears in § 4 of the Appendix. Proofs of these theorems can be found in most standard works in probability theory and real analysis.

In writing the present book I am indebted to Doob [2] and Itô [2] from which I taught in the past. The influence of these two works are evident in many parts of the present book. I take this opportunity to express my gratitude to Professor Robert H. Cameron from whom I learned integration in functions spaces. Drs. William Hudson, Richard Brooks, Edward Kerlin, Martin Walter and Wesley Masenten contributed in improving the text. For the expert typing of the text I thank Lillian White, Linda Husak and Grace Koo the first of whom typed most of the manuscript. My thanks are due to the editorial board of the series Pure and Applied Mathematics, and in particular Professor Howard G. Tucker, for inviting me to write this book.

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CONTENTS

Chapter 1

STOCHASTIC PROCESSES

§1. STOCHASTIC PROCESSES

A stochastic process is by definition a collection of random variables $\{X_t, t \in D\}$ defined on a probability space $(\Omega, \mathfrak{B}, P)$ where the index set D is a subset of the real line R^1. Thus a stochastic process X is a real-valued function $X(t, \omega)$ on $D \times \Omega$ which is a $\mathfrak{B}$-measurable function on Ω for each $t \in D$. We shall occasionally use the notation $X(t)$ to mean the random variable $X(t, \cdot)$. The index set D is called the domain of definition of the stochastic process. For each $\omega \in \Omega$, the real-valued function $X(\cdot, \omega)$ is called a sample function or a sample path of the stochastic process. Sometimes it is necessary to permit some of the random variables $X(t, \cdot)$, $t \in D$, to assume extended real values, and then we speak of an extended real-valued stochastic process. Otherwise a stochastic process is always real valued.

Definition 1.1 Two stochastic processes X and Y on a probability space $(\Omega, \mathfrak{B}, P)$ and $D \subset R^1$ are said to be equivalent if, for every $t \in D$, $X(t, \omega) = Y(t, \omega)$ for a.e. ω, i.e., there exists $\Lambda_t \in \mathfrak{B}$ with $P(\Lambda_t) = 0$ such that $X(t, \omega) = Y(t, \omega)$ for $\omega \in \Lambda_t^c$. X and Y are said to be almost surely equal if the sample functions of X and Y are identical for a.e. ω, i.e., there exists $\Lambda \in \mathfrak{B}$ with $P(\Lambda) = 0$ such that $X(\cdot, \omega) = Y(\cdot, \omega)$ for $\omega \in \Lambda^c$.

Definition 1.2 A stochastic process X on a probability space $(\Omega, \mathfrak{B}, P)$ and $D \subset R^1$ is said to be almost surely continuous if a.e. sample function is continuous on D, i.e., there exists

$\Lambda \in \mathfrak{B}$ with $P(\Lambda) = 0$ such that $X(\cdot, \omega)$ is a continuous function on D for $\omega \in \Lambda^c$.

Remark 1.3 Two almost surely continuous stochastic processes which are equivalent are in fact almost surely equal.

Proof Let X and Y be almost surely continuous and equivalent stochastic processes on $(\Omega, \mathfrak{B}, P)$ and D. Let $\Lambda \in \mathfrak{B}$ with $P(\Lambda) = 0$ be such that $X(\cdot, \omega)$ and $Y(\cdot, \omega)$ are continuous on D for $\omega \in \Lambda^c$. Let $S = \{s_n, n = 1, 2, \cdots\}$ be a countable dense subset of D whose elements are numbered in an arbitrary way. By the equivalence of X and Y, for every n there exists $\Lambda_n \in \mathfrak{B}$ with $P(\Lambda_n) = 0$ such that

$$X(s_n, \omega) = Y(s_n, \omega) \quad \text{for} \quad \omega \in \Lambda_n^{\ c}.$$

Let $\Lambda_\infty = \bigcup_{n=1}^{\infty} \Lambda_n$. Then $\Lambda_\infty \in \mathfrak{B}$ with $P(\Lambda_\infty) = 0$ and

$$X(s_n, \omega) = Y(s_n, \omega) \quad \text{for} \quad n = 1, 2, \cdots \quad \text{when} \quad \omega \in \Lambda_\infty^{\ c}.$$

On the other hand for an arbitrary $t \in D$ and $\omega \in \Lambda^c$, $X(t, \omega) = \lim_{k\to\infty} X(s_{n_k}, \omega)$ and $Y(t, \omega) = \lim_{k\to\infty} Y(s_{n_k}, \omega)$ for every subsequence $\{s_{n_k}\} \subset \{s_n\}$ such that $\lim_{k\to\infty} s_{n_k} = t$. Then from the fact that $X(s_{n_k}, \omega) = Y(s_{n_k}, \omega)$, $k = 1, 2, \cdots$, for $\omega \in \Lambda_\infty^{\ c}$ we have $X(t, \omega) = Y(t, \omega)$ for $\omega \in (\Lambda \cup \Lambda_\infty)^c$. □

Definition 1.4 Given a stochastic process X on a probability space $(\Omega, \mathfrak{B}, P)$ and $D \subset R^1$. Let $T = \{t_1, t_2, \cdots, t_n\}$ be a finite sequence of distinct elements of D. Consider the transformation $X_T = X_{t_1 \cdots t_n} = \left(X(t_1, \cdot), \cdots, X(t_n, \cdot)\right)$ of Ω into R^n and let $\Phi_{X_T} = \Phi_{X_{t_1 \cdots t_n}}$ be the n-dimensional probability distribution determined by X_T, i.e., the probability measure on the σ-field of

Borel sets in R^n, $\mathfrak{B}^n$, defined by

$$\Phi_{X_T}(E) = P\left(X_T^{-1}(E)\right) \text{ for } E \in \mathfrak{B}^n.$$

Let $\mathfrak{T}$ be the collection of all the T's. Then $\{\Phi_{X_T}, T \in \mathfrak{T}\}$ is called the system of finite dimensional probability distributions determined by X.

Remark 1.5 For two equivalent stochastic processes X and Y on a probability space $(\Omega, \mathfrak{B}, P)$ and $D \subset R^1$ we have $\Phi_{X_T} = \Phi_{Y_T}$ for every $T \in \mathfrak{T}$.

Proof Let $T = \{t_1, t_2, \cdots, t_n\}$. Since a probability measure on $\mathfrak{B}^n$ is uniquely determined by its values for intervals in R^n of the type $I = (\alpha_1, b_1] \times \cdots \times (\alpha_n, b_n]$, to show that $\Phi_{X_T} = \Phi_{Y_T}$ it suffices to show that $\Phi_{X_T}(I) = \Phi_{Y_T}(I)$ for every I. Now at each t_k, $k = 1, 2, \cdots, n$, from the equivalence of X and Y there exists $\Lambda_k \in \mathfrak{B}$ with $P(\Lambda_k) = 0$ such that $X(t_k, \omega) = Y(t_k, \omega)$ when $\omega \in \Lambda_k^c$. Let $\Lambda = \bigcup_{k=1}^n \Lambda_k$. Then $P(\Lambda) = 0$ so that

$$\begin{aligned}
\Phi_{X_T}(I) &= P\left\{\omega \in \Omega\,;\, X(t_k, \omega) \in (\alpha_k, b_k],\ k = 1, 2, \cdots, n\right\} \\
&= P\left\{\omega \in \Lambda^c\,;\, X(t_k, \omega) \in (\alpha_k, b_k],\ k = 1, 2, \cdots, n\right\} \\
&= P\left\{\omega \in \Lambda^c\,;\, Y(t_k, \omega) \in (\alpha_k, b_k],\ k = 1, 2, \cdots, n\right\} \\
&= P\left\{\omega \in \Omega\,;\, Y(t_k, \omega) \in (\alpha_k, b_k],\ k = 1, 2, \cdots, n\right\} \\
&= \Phi_{Y_T}(I).
\end{aligned}$$

□

§2. EXISTENCE OF A STOCHASTIC PROCESS WITH A GIVEN SYSTEM OF FINITE DIMENSIONAL PROBABILITY DISTRIBUTIONS

In §1 we saw that a stochastic process X on a probability space $(\Omega, \mathfrak{B}, P)$ and $D \subset R^1$ determines a system of finite dimensional probability distributions $\{\Phi_{X_T}, T \in \mathfrak{T}\}$ where $\mathfrak{T}$ is the collection of all finite sequences of distinct elements $T = \{t_1, t_2, \cdots, t_n\}$ from D. We also saw that if X and Y are two equivalent stochastic processes on $(\Omega, \mathfrak{B}, P)$ and D then $\Phi_{X_T} = \Phi_{Y_T}$ for every $T \in \mathfrak{T}$. It is natural to ask the question: Given a system of finite dimensional probability distributions $\{\Phi_T, T \in \mathfrak{T}\}$ does a stochastic process X on some probability space $(\Omega, \mathfrak{B}, P)$ and D exist which has $\{\Phi_T, T \in \mathfrak{T}\}$ as its system of finite dimensional probability distributions, that is, $\Phi_{X_T} = \Phi_T$ for every $T \in \mathfrak{T}$? The answer to this question is in the affirmative provided that the system $\{\Phi_T, T \in \mathfrak{T}\}$ satisfies certain consistency conditions. To prove this we need the Kolmogorov Extension Theorem.

[I] The Kolmogorov Extension Theorem

Given an arbitrary set A and the collection R^A of all real-valued functions defined on A. An element of R^A is denoted by $\omega = \left(\omega(\alpha), \alpha \in A\right)$ and the real number $\omega(\alpha)$ is called the α-coordinate of ω. For a finite sequence of distinct elements of A, $\{\alpha_1, \cdots, \alpha_n\}$, the projection of R^A onto R^n with index $\{\alpha_1, \cdots, \alpha_n\}$, namely, $p_{\alpha_1 \cdots \alpha_n}$, is defined by

$$p_{\alpha_1 \cdots \alpha_n}(\omega) = \left(\omega(\alpha_1), \cdots, \omega(\alpha_n)\right) \in R^n \text{ for every } \omega \in R^A.$$

The Borel cylinder in R^A with index $\{\alpha_1, \cdots, \alpha_n\}$ and base $B \in \mathfrak{B}^n$ is a subset of R^A defined by

$$p_{\alpha_1 \cdots \alpha_n}^{-1}(B) = \left\{\omega \in R^A ;\, p_{\alpha_1 \cdots \alpha_n}(\omega) \in B\right\}.$$

Let $\mathfrak{Z}_{\alpha_1 \cdots \alpha_n}$ be the collection of all Borel cylinders in R^A with index $\{\alpha_1, \cdots, \alpha_n\}$, i.e.,

$$\mathfrak{F}_{\alpha_1 \cdots \alpha_n} = p_{\alpha_1 \cdots \alpha_n}^{-1}(\mathfrak{B}^n) = \left\{ p_{\alpha_1 \cdots \alpha_n}^{-1}(B),\ B \in \mathfrak{B}^n \right\}$$

and let

$$\mathfrak{F} = \bigcup \mathfrak{F}_{\alpha_1 \cdots \alpha_n}$$

where the union is over all finite sequences of distinct elements of A. As we shall show in Remark 2.2, $\mathfrak{F}_{\alpha_1 \cdots \alpha_n}$ is a σ-field and $\mathfrak{F}$ is a field of subsets of R^A.

Theorem 2.1 Kolmogorov Extension Theorem. Let μ be a set function defined on the field $\mathfrak{F} = \bigcup \mathfrak{F}_{\alpha_1 \cdots \alpha_n}$ of subsets of R^A satisfying the condition that, for every $\{\alpha_1, \cdots, \alpha_n\}$, μ is a probability measure on the σ-field $\mathfrak{F}_{\alpha_1 \cdots \alpha_n}$. Then μ can be extended uniquely to be a probability measure on the σ-field $\sigma(\mathfrak{F})$ generated by $\mathfrak{F}$.

Before we prove Theorem 2.1 let us enumerate some pertinent properties of $\mathfrak{F}_{\alpha_1 \cdots \alpha_n}$ and $\mathfrak{F}$ in the form of a remark.

Remark 2.2 1°. $\mathfrak{F}_{\alpha_1 \cdots \alpha_n}$ is a σ-field of subsets of R^A.

Proof (1) $R^A = p_{\alpha_1 \cdots \alpha_n}^{-1}(R^n)$. Thus $R^A \in \mathfrak{F}_{\alpha_1 \cdots \alpha_n}$.

(2) Let $E \in \mathfrak{F}_{\alpha_1 \cdots \alpha_n}$. Then $E = p_{\alpha_1 \cdots \alpha_n}^{-1}(B)$ for some $B \in \mathfrak{B}^n$ so that $E^c = p_{\alpha_1 \cdots \alpha_n}^{-1}(B^c) \in \mathfrak{F}_{\alpha_1 \cdots \alpha_n}$.

(3) Let $\{E_i, i = 1, 2, \cdots\} \subset \mathfrak{F}_{\alpha_1 \cdots \alpha_n}$. Then $E_i = p_{\alpha_1 \cdots \alpha_n}^{-1}(B_i)$ for some $B_i \in \mathfrak{B}^n$ so that

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} p_{\alpha_1 \cdots \alpha_n}^{-1}(B_i) = p_{\alpha_1 \cdots \alpha_n}^{-1}\left(\bigcup_{n=1}^{\infty} B_i\right) \in \mathfrak{F}_{\alpha_1 \cdots \alpha_n}. \quad \square$$

2°. If $E \in \mathfrak{F}_{\alpha_1 \cdots \alpha_n}$ then by the definition of $\mathfrak{F}_{\alpha_1 \cdots \alpha_n}$ there exists some $B \in \mathfrak{B}^n$ such that $E = p_{\alpha_1 \cdots \alpha_n}^{-1}(B)$. The set B is uniquely determined by E, i.e., if $B_1, B_2 \in \mathfrak{B}^n$, $B_1 \neq B_2$, then $p_{\alpha_1 \cdots \alpha_n}^{-1}(B_1) \neq p_{\alpha_1 \cdots \alpha_n}^{-1}(B_2)$ or, equivalently, if $E_1, E_2 \in \mathfrak{F}_{\alpha_1 \cdots \alpha_n}$ and $E_1 = p_{\alpha_1 \cdots \alpha_n}^{-1}(B_1)$, $E_2 = p_{\alpha_1 \cdots \alpha_n}^{-1}(B_2)$ with $B_1, B_2 \in \mathfrak{B}^n$ then $E_1 = E_2$ implies $B_1 = B_2$. This is from the fact that $p_{\alpha_1 \cdots \alpha_n}(E_1) = B_1$, $p_{\alpha_n \cdots \alpha_n}(E_2) = B_2$ so that $E_1 = E_2$ implies $B_1 = B_2$.

3°. Consider $\mathfrak{F} = \bigcup \mathfrak{F}_{\alpha_1 \cdots \alpha_n}$. If $E \in \mathfrak{F}$ then $E \in \mathfrak{F}_{\alpha_1 \cdots \alpha_n}$ for some index $\{\alpha_1, \cdots, \alpha_n\}$ and $E = p_{\alpha_1 \cdots \alpha_n}^{-1}(B)$ for some $B \in \mathfrak{B}^n$ which is uniquely determined as long as $\{\alpha_1, \cdots, \alpha_n\}$ is fixed according to 2°. However E may belong to other σ-fields than $\mathfrak{F}_{\alpha_1 \cdots \alpha_n}$. For instance if $E = p_{\alpha_1 \cdots \alpha_n}^{-1}(B)$ then certainly $E = p_{\alpha_1 \cdots \alpha_{n+1}}^{-1}(B \times R^1)$ with an arbitrary $\alpha_{n+1} \in A$, $\alpha_{n+1} \neq \alpha_1, \cdots, \alpha_n$, so that $E \in \mathfrak{F}_{\alpha_1 \cdots \alpha_n \alpha_{n+1}}$.

Also if $\Pi = \{\Pi_1, \cdots, \Pi_n\}$ is a permutation of $\{1, \cdots, n\}$ and $\Pi(B) \in \mathfrak{B}^n$ is the permuted set of $B \in \mathfrak{B}^n$ by Π, i.e., the set in R^n traced by $(\xi_{\Pi_1} \cdots \xi_{\Pi_n}) \in R^n$ when $(\xi_1, \cdots, \xi_n) \in R^n$ traces B, then certainly $E = p_{\alpha_1 \cdots \alpha_n}^{-1}(B) = p_{\alpha_{\Pi_1} \cdots \alpha_{\Pi_n}}^{-1}\left(\Pi(B)\right)$ so that

$E \in \mathfrak{F}_{\alpha_{\Pi_1} \cdots \alpha_{\Pi_n}}$ also. Note also that $\mathfrak{F}_{\alpha_1 \cdots \alpha_n} = \mathfrak{F}_{\alpha_{\Pi_1} \cdots \alpha_{\Pi_n}}$.

4°. Let $E \in \mathfrak{F}$. As we saw in 3°, E has more than one representation by finite dimensional Borel sets. A representation $E = p_{\alpha_1 \cdots \alpha_n}^{-1}(B)$, $B \in \mathfrak{B}^n$, is called a minimal representation for E if B is not the Cartesian product of R^1 with an $n-1$ dimensional Borel set. The space R^A, for any index $\{\alpha_1, \cdots, \alpha_n\}$, being representable as $p_{\alpha_1 \cdots \alpha_n}^{-1}(R^n)$ only, has no minimal representation. For any $E \in \mathfrak{F}$, $E \neq R^A$, any representation $E = p_{\alpha_1 \cdots \alpha_n}^{-1}(B)$, $B \in \mathfrak{B}^n$, which is not already a minimal representation can be reduced to a minimal representation since E can also be given as $E = p_{\alpha_{\Pi_1} \cdots \alpha_{\Pi_n}}^{-1}\left(\Pi(B)\right)$ where $\Pi(B) = B' \times R^1$ with an $n-1$ dimensional Borel set B' and consequently $E = p_{\alpha_{\Pi_1} \cdots \alpha_{\Pi_{n-1}}}^{-1}(B')$. If this last representation is not yet a minimal representation we repeat the process of reduction until we arrive at a minimal representation.

5°. Let $E \in \mathfrak{F}$, $E \neq R^A$, and let $p_{\alpha_1 \cdots \alpha_n}^{-1}(B_1)$, $B_1 \in \mathfrak{B}^n$, $p_{\beta_1 \cdots \beta_m}^{-1}(B_2)$, $B_2 \in \mathfrak{B}^m$, be two minimal representations of E. Then $m = n$ and $\{\beta_1, \cdots, \beta_m\} = \{\alpha_{\Pi_1}, \cdots, \alpha_{\Pi_n}\}$ for some permutation $\Pi = \{\Pi_{\alpha_1}, \cdots, \Pi_{\alpha_n}\}$ of $\{1, \cdots, n\}$ and $B_2 = \Pi(B_1)$.

<u>Proof</u> (1) To show $m = n$, assume for instance $n < m$. Then for some j_0 we have $\beta_{j_0} \neq \alpha_1, \cdots \alpha_n$. Then B_2 must be the Cartesian product of R^1 and an $m-1$ dimensional Borel set contradicting the minimality of the representation $p_{\beta_1 \cdots \beta_m}^{-1}(B_2)$.

Thus $n \geq m$. Similarly we have $m \geq n$ and hence $m = n$.

(2) The argument in (1) shows not only that $m = n$ but also that as two point sets $\{\alpha_1, \cdots, \alpha_n\}$ and $\{\beta_1, \cdots, \beta_n\}$ are equal.

(3) Now that $\{\alpha_1, \cdots, \alpha_n\}$ and $\{\beta_1, \cdots, \beta_n\}$ are equal as point sets, there exists a permutation $\Pi = \{\Pi_1, \cdots, \Pi_n\}$ of $\{1, \cdots, n\}$ so that $\{\beta_1, \cdots, \beta_n\} = \{\alpha_{\Pi_1}, \cdots, \alpha_{\Pi_n}\}$. Then

$$E = p_{\alpha_1 \cdots \alpha_n}^{-1}(B_1) = p_{\alpha_{\Pi_1} \cdots \alpha_{\Pi_n}}^{-1}\left(\Pi(B_1)\right) = p_{\beta_1 \cdots \beta_n}^{-1}\left(\Pi(B_1)\right).$$

But $E = p_{\beta_1 \cdots \beta_n}^{-1}(B_2)$ also. Thus by 2°, $B_2 = \Pi(B_1)$. □

6°. $\mathfrak{F}$ is a field of subsets of R^A.

<u>Proof</u> (1) $R^A \in \mathfrak{F}$. In fact $R^A = p_{\alpha_1 \cdots \alpha_n}^{-1}(R^n) \in \mathfrak{F}_{\alpha_1 \cdots \alpha_n}$ for every $\{\alpha_1, \cdots, \alpha_n\}$.

(2) If $E \in \mathfrak{F}$ then $E \in \mathfrak{F}_{\alpha_1 \cdots \alpha_n}$ for some $\{\alpha_1, \cdots, \alpha_n\}$ and hence $E^c \in \mathfrak{F}_{\alpha_1 \cdots \alpha_n} \subset \mathfrak{F}$ since $\mathfrak{F}_{\alpha_1 \cdots \alpha_n}$ is a σ-field.

(3) Let $E_1, E_2 \in \mathfrak{F}$. Then $E_1 = p_{\alpha_1 \cdots \alpha_n}^{-1}(B_1)$, $B_1 \in \mathfrak{B}^n$, and $E_2 = p_{\beta_1 \cdots \beta_m}^{-1}(B_2)$, $B_2 \in \mathfrak{B}^m$. If $\{\alpha_1 \cdots \alpha_n\}$ and $\{\beta_1, \cdots, \beta_m\}$ are disjoint as point sets then we have

$$E_1 = p_{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m}^{-1}(B_1 \times R^m), \quad E_2 = p_{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m}^{-1}(R^n \times B_2)$$

and hence

$$E_1 \cup E_2 = p_{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m}^{-1}(B_1 \times R^m \cup R^n \times B_2) \in \mathfrak{F}_{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m} \subset \mathfrak{F}.$$

If $\{\alpha_1, \cdots, \alpha_n\}$ and $\{\beta_1, \cdots, \beta_m\}$ are not disjoint as point sets then permute $\{\alpha_1, \cdots, \alpha_n\}$, $\{\beta_1, \cdots, \beta_m\}$, B_1 and B_2 if necessary so that

$$\alpha_i = \beta_i \text{ for } i = 1, 2, \cdots, \ell \text{ and } \alpha_i \neq \beta_j \text{ for } i = \ell+1, \cdots, n, j = \ell+1, \cdots, m.$$

Then

$$E_1 = p_{\alpha_1 \cdots \alpha_n}^{-1}(B_1) = p_{\alpha_1 \cdots \alpha_\ell \, \alpha_{\ell+1} \cdots \alpha_n \, \beta_{\ell+1} \cdots \beta_m}^{-1}(B_1 \times R^{m-\ell})$$

$$E_2 = p_{\beta_1 \cdots \beta_m}^{-1}(B_2) = p_{\alpha_1 \cdots \alpha_\ell \, \beta_{\ell+1} \cdots \beta_m \, \alpha_{\ell+1} \cdots \alpha_n}^{-1}(B_2 \times R^{n-\ell})$$

$$= p_{\alpha_1 \cdots \alpha_\ell \, \alpha_{\ell+1} \cdots \alpha_n \, \beta_{\ell+1} \cdots \beta_m}^{-1}\left(\Pi(B_2 \times R^{n-\ell})\right)$$

with a suitable permutation Π. Then $E_1 \cup E_2 \in \mathfrak{F}_{\alpha_1 \cdots \alpha_n \beta_{\ell+1} \cdots \beta_m} \subset \mathfrak{F}$. □

Lemma 2.3 Given a k-dimensional probability space $(R^k, \mathfrak{B}^k, \Phi)$; For every $A \in \mathfrak{B}^k$ and $\epsilon > 0$ there exist compact sets C and D such that $C \subset A$, $D \subset A^c$, and $\Phi(A - C)$, $\Phi(A^c - D) < \epsilon$.

Proof Let $\mathfrak{K}$ be the collection of members of $\mathfrak{B}^k$ for which the statement of the lemma holds. Let $\mathfrak{I}^k$ be the collection of subsets of R^k of the type $(a_1, b_1] \times \cdots \times (a_k, b_k]$ which generates the σ-field $\mathfrak{B}^k$. If we show that $\mathfrak{I}^k \subset \mathfrak{K}$ and that $\mathfrak{K}$ is a σ-field then $\mathfrak{B}^k \subset \mathfrak{K}$, and the proof is complete.

We show first that $\mathfrak{I}^k \subset \mathfrak{K}$. Now if $A \in \mathfrak{I}^k$ then clearly both A and A^c are limits of monotone increasing sequences of compact sets $\{C_n, n = 1, 2, \cdots\}$ and $\{D_n, n = 1, 2, \cdots\}$ so that, for sufficiently large n, we have $\Phi(A - C_n)$, $\Phi(A^c - D_n) < \epsilon$ and $A \in \mathfrak{K}$.

Next we show that $\mathfrak{K}$ is a σ-field. First of all, $R^k \in \mathfrak{I}^k \subset \mathfrak{K}$. Secondly by the definition of $\mathfrak{K}$, $A \in \mathfrak{K}$ implies $A^c \in \mathfrak{K}$. Finally let