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FOREWORD

The Transactions of the American Society of Civil Engineers, Volume 125, 1960, consists of two parts, of which this is Part I. Volume 125, Part II (which is available only in cloth binding) is a Symposium on Rockfill Dams, the "core" of which is a series of papers presented before the Power Division of ASCE in June 1958 at Portland, Oregon. The decision to issue the second part to Volume 125 was made by the ASCE Committee on Publications in June 1960, in Reno, Nevada, when it became apparent that the interest in this subject warranted the publication of a volume of Transactions devoted exclusively to the science of rockfill dams.

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DIGITAL COMPUTATION FOR STIFFNESS MATRIX ANALYSIS

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SYNOPSIS

An energy development of the basic theory for the stiffness matrix method of structural analysis is contained herein. The method is particularly adaptable to a systematic analysis of any structure which may be described as a composition of simple structural elements, such as beams and plates.

Application of the technique to the analysis of structures is presented, with an example to show how the procedure works in a typical case. The present application of the technique to the analysis of aircraft structures is also described. Some suggestions are given for a practical method of developing a similar procedure for individual needs.

INTRODUCTION

History.—The stiffness matrix type of structural analysis was described in 1944 by Gabriel Kron.[2] The first practical application of the technique using digital computers was reported by H. U. Schuerch[3] in December 1951. Samuel Levy[4] investigated the stiffness matrix procedure which was published in July and September 1953. Gilbert Best and M. Phillip Keating[5] described the technique based on their experience with the IBM 701. D. Williams[6] prepared a

Note.—Published essentially as printed here, in October, 1958, in the Journal of the Structural Division, as Proceedings Paper 1814. Positions and titles given are those in effect when the paper was approved for publication in Transactions.

[1] Project Structures Engr., Convair, General Dynamics Corp., Fort Worth, Tex.

[2] "Tensorial Analysis and Equivalent Circuits of Elastic Structures," by G. Kron, Journal, Franklin Inst., Vol. 238, No. 6, 1944.

[3] "Vibration Mode Analysis For Delta Wing Structure," by H. U. Schuerch, Convair Memo DG-G-100, San Diego, December 18, 1951.

[4] "Structural Analysis and Influence Coefficients for Delta Wings," by S. Levy, Journal of the Aeronautical Sciences, Vol. 20, No. 7, July, 1953.

[5] "A Stiffness Matrix Method of Delta Wing Stress Analysis," by G. Best and M. P. Keating, Convair SRG-16, Fort Worth, Tex., September 8, 1953.

[6] "Recent Developments In the Structural Approach to Aeroelastic Problems," by D. Williams, Journal of the Royal Aeronautical Society, London, Eng., June, 1954.

comprehensive paper on the subject. The following year the technique of using matrix algebra for the analysis of aircraft structures was published by Raymond L. Bisplinghoff.[7]

Theory.—Let U represent the strain energy stored within a structure that is loaded by the forces $P_1, P_2, \ldots, P_i, \ldots, P_n$. The temperature of the material remains constant and the supports are rigid. Applying Castigliano's theorem of structure equilibrium (Theorem I):

$$P_i = \frac{\partial U}{\partial \delta_i} \quad \ldots\ldots\ldots\ldots (1)$$

in which δ_i is the deflection of the point of application of the load P_i in the direction of P_i.

If the strain energy is evaluated in terms of the loads P_i acting upon the structure we may expand Eq. 1 as follows:

$$P_i = \frac{\partial U}{\partial \delta_i} = \sum_j \left(\frac{\partial U}{\partial P_j}\right)\left(\frac{\partial P_j}{\partial \delta_i}\right) \quad \ldots\ldots\ldots\ldots (2)$$

If the structure is assumed to be elastic, then Castigliano's theorem for linear structures (Theorem II) may be applied:

$$\delta_j = \frac{\partial U}{\partial P_j} \quad \ldots\ldots\ldots\ldots (3)$$

Substituting Eq. 3 into Eq. 2,

$$P_i = \sum_j \delta_j \left(\frac{\partial P_j}{\partial \delta_i}\right) \quad \ldots\ldots\ldots\ldots (4)$$

The partial derivative $\frac{\partial P_j}{\partial \delta_i}$ represents the force developed at point j due to a deflection of point i, all other points remaining fixed. This force is represented by the symbol s_{ji}, the subscript j representing the point at which the force acts, and the subscript i representing the point at which the unit deflection is imposed. With this substitution, Eq. 4 becomes:

$$P_i = \sum_j \delta_j s_{ji} \quad \ldots\ldots\ldots\ldots (5)$$

From Betti's Law or the generalized Maxwells' law of reciprocal deflections we obtain the relation

$$s_{ji} = s_{ij} \quad \ldots\ldots\ldots\ldots (6)$$

and, hence,

$$P_i = \sum_j \delta_j s_{ij} \quad \ldots\ldots\ldots\ldots (7)$$

Writing Eq. 7 in its expanded form,

$$P_i = s_{i1}\delta_1 + s_{i2}\delta_2 + \ldots + s_{ij}\delta_j + \ldots + s_{in}\delta_n \quad \ldots\ldots (8)$$

[7] "Aeroelasticity," by R. L. Bisplinghoff, H. Ashley and R. L. Halfmann, Addison-Wesley Publishing Co. Inc., Cambridge, Mass., 1955.

It is evident from the expanded form that Eq. 7 is a superposition equation expressing the total load at point i as the sum of the loads developed by each deflection component δ_j acting by itself. Each part of Eq. 7 describes an independent component of the structural behavior. The components may represent translation or rotation. The total number of components is the number of degrees of freedom that the idealized structure possesses.

Using matrix algebra notation, Eq. 7 may be rewritten as

$$P = S\,\delta \qquad (9)$$

in which P is a vector or column matrix made up of load components P_1, P_2, . . ., P_i, . . ., P_n; and δ is a vector made up of the deflection components δ_1, δ_2, . . ., δ_i, . . . δ_n, and S is a square matrix consisting of an ordered array of the stiffness influence coefficients s_{ij} of Eq. 7. Matrix S is called the stiffness matrix of the structure. In the expanded form, Eq. 9 appears as follows:

$$\begin{Bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \\ P_i \\ \cdot \\ \cdot \\ \cdot \\ P_n \end{Bmatrix} = \begin{bmatrix} s_{11} & s_{12} \cdots & s_{1j} \cdots & s_{1n} \\ s_{21} & s_{22} & s_{2j} & s_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ s_{i1} & s_{i2} & s_{ij} & s_{in} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ s_{n1} & s_{n2} \cdots & s_{nj} \cdots & s_{nn} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \cdot \\ \cdot \\ \cdot \\ \delta_j \\ \cdot \\ \cdot \\ \cdot \\ \delta_n \end{Bmatrix} \qquad (10)$$

Eq. 9 may be manipulated similar to an ordinary algebraic equation. For instance, it may be solved for the deflections by premultiplying both sides of the equation by the inverse of the stiffness matrix S.

$$\delta = S^{-1} P = F P \qquad (11)$$

The symbol F represents the inverse of the stiffness matrix which in turn represents the flexibility influence coefficient matrix. It is composed of an ordered array of the flexibility influence coefficients f_{ij}. In the expanded form it appears as follows:

$$\begin{Bmatrix} \delta_1 \\ \delta_2 \\ \cdot \\ \cdot \\ \cdot \\ \delta_i \\ \cdot \\ \cdot \\ \cdot \\ \delta_n \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1j} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2j} & \cdots & f_{2n} \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ f_{i1} & f_{i2} & \cdots & f_{ij} & \cdots & f_{in} \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ f_{n1} & f_{n2} & \cdots & f_{nj} & \cdots & f_{nn} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \\ P_j \\ \cdot \\ \cdot \\ \cdot \\ P_n \end{Bmatrix} \quad \ldots (12)$$

In a typical problem, Eq. 11 must be constructed and solved for the deflections in terms of the loads. The direct determination of the coefficients f_{ij} of Eq. 11 is difficult and impractical for a large indeterminate structure. However, the coefficients s_{ij} of Eq. 9 are readily calculated. The usual procedure is to compute the stiffness matrix coefficients directly from the known properties of the individual elements of the idealized structure. The coefficients of the flexibility matrix are obtained through the mathematical operation of inverting the stiffness matrix.

After the deflections of the structure are found, the stresses developed in the structure may be determined. This is accomplished by using the stiffness matrix of the individual elements of the structure. Using computed deflections and equations of the same type as Eq. 9, the internal loads on individual parts of the structure are obtained readily. After the internal forces on the individual elements have been obtained, the stresses in the structure can be readily computed.

PRESENT APPLICATIONS

Structure analyses have been performed on the IBM 701 for structures having up to 106 independent coordinates. The structures analyzed were idealized as a group of interconnected structural elements. These elements consisted of beams, torque tubes, torque boxes, and concentrated springs (Fig. 1). The procedure followed for a typical stress analysis was as follows:

1. Compute the stiffness matrices for the individual structural elements (Fig. 2(a)). Store the results on magnetic tape.
2. Assemble the stiffness matrix for the complete structure by bringing together the coefficients for all structural elements and adding corresponding coefficients (Fig. 2(b)). Individual coordinate identification numbers are indispensible for keeping the coefficients in order.
3. Invert the stiffness matrix to obtain the flexibility matrix for the structure ($F = S^{-1}$).
4. Compute the deflections by determining the matrix product of the applied load vector and the flexibility matrix ($\delta = F\,P$).

5. Compute the internal forces on the individual structural elements by computing the product of the appropriate deflections with the stiffness matrices for the individual structural elements.

$$\begin{Bmatrix} P_{2A} \\ P_{3A} \end{Bmatrix} = [A] \begin{Bmatrix} \delta_2 \\ \delta_3 \end{Bmatrix} \qquad (13a)$$

$$\begin{Bmatrix} P_{3B} \\ P_{4B} \end{Bmatrix} = [B] \begin{Bmatrix} \delta_3 \\ \delta_4 \end{Bmatrix} \qquad (13b)$$

$$\begin{Bmatrix} P_{1C} \\ P_{2C} \\ P_{5C} \\ P_{6C} \end{Bmatrix} = [C] \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_5 \\ \delta_6 \end{Bmatrix} \qquad (13c)$$

and

$$\begin{Bmatrix} P_{6D} \\ P_{7D} \end{Bmatrix} = [D] \begin{Bmatrix} \delta_6 \\ \delta_7 \end{Bmatrix} \qquad (13d)$$

The largest problems, of the order of 106 independent coordinates, required a total of 4 hr to assemble the stiffness matrix and 8 hr to invert in order to obtain the flexibility matrix. Double precision arithmetic was used in this operation. Each stress calculation for a given load condition was obtained in an additional $1\frac{1}{2}$ hr of computing time. Smaller problems were solved in considerably less time.

The analysis of beam elements was performed considering bending deformation only. A linear variation in moment of inertia from one end of a beam to the other was assumed. In addition to the stress analyses, normal mode analyses were successfully performed. In such an analysis, the natural frequencies and normal mode shapes were computed by an iteration technique.

The IBM 704 is much faster and has a larger speed storage capacity than the 701 computer. Therefore, a substantial savings in time can be realized by its use. Experience indicates that a matrix representing 100 independent coordinates can be assembled in 1 hr and inverted in 3 hr.

The 704 computer program has incorporated the following additions and refinements to the aforementioned 701 computer program:

1. Shear deformation is included in the analysis of beams.
2. Effect of tapering flanges is computed.
3. Rectangular structural plates may be analyzed using a lattice analogy representation.
4. Beam moments of inertia are automatically computed if desired.
5. Length and direction of beams are determined by the computer from basic geometric input.

The stiffness matrix structural analysis has been coded on the Remington Rand 1103, a computer comparable to the IBM 701. The sizes of the problems worked by the 1103 computer were generally smaller than those worked by the 701 computer. However, the speed with which a given problem could be worked was greater for the 1103.

APPLICATION TO STRUCTURES

Indeterminate Trusses.—A bridge truss could be analyzed easily by the stiffness matrix technique, but because the analysis of a determinate truss problem is simple, it would not be worthwhile to analyze it on a high-speed

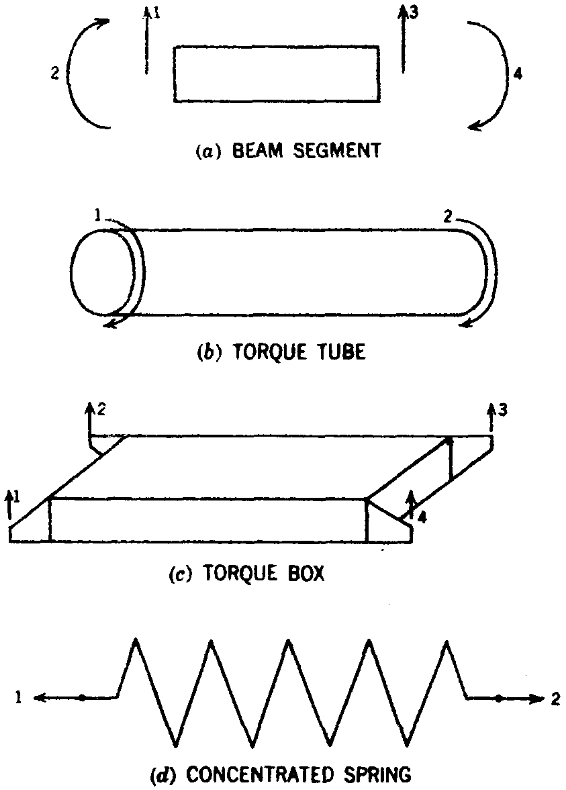


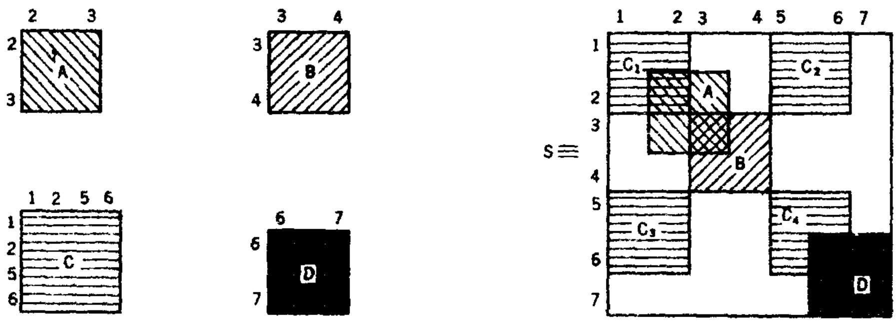
FIG. 1.—INDIVIDUAL ELEMENT STIFFNESS MATRIX

digital computer. This is not always the case in the design of an indeterminate truss.

Consider the truss in Fig. 3(a). The distortion and loads on this structure can be described by two orthogonal displacements at each joint. Hence, the analysis of the structure requires a matrix of order 17 for its solution. If a secondary stress analysis is required, it may be obtained simultaneously with the primary analysis. However, a rotation displacement at each joint must be represented, requiring a matrix of order 27 for its solution (Fig. 3(b)). The stress analysis of the truss gives the forces in all members and the deflections of all joints. For a structure of this type the order of the matrix is generally much greater than the degree to which the structure is indeterminate.

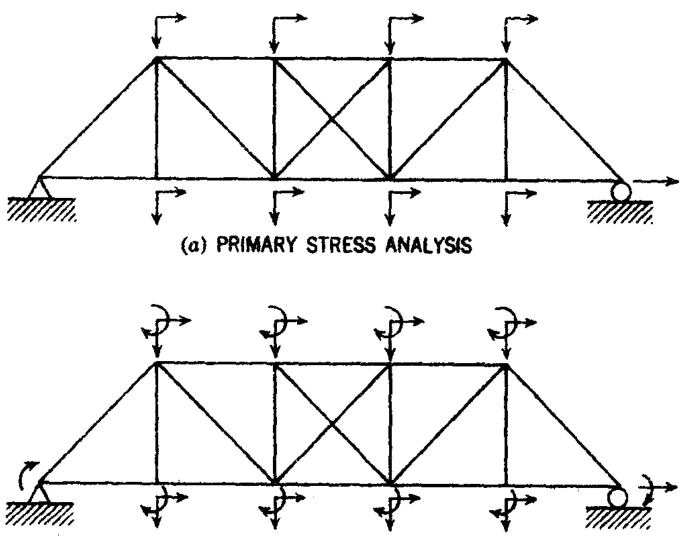
Building Frames.—For the simple building bent (Fig. 4(a)) the distortion is completely defined by five coordinates: One rotation at each joint, and one displacement at the roof level. Even in the case of a more complicated broken story bent (Fig. 4(b)) only twelve coordinates are necessary.

It is evident from these examples that the analysis of building frames could be accomplished by using the stiffness matrix technique of analysis. If three-



(a) COMPUTE (b) ASSEMBLE

FIG. 2.—STIFFNESS MATRIX PROCEDURE



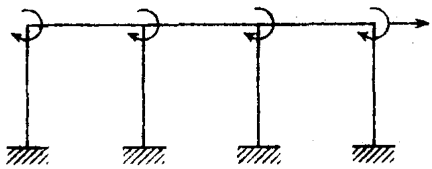
(a) PRIMARY STRESS ANALYSIS

(b) PRIMARY AND SECONDARY STRESS ANALYSIS

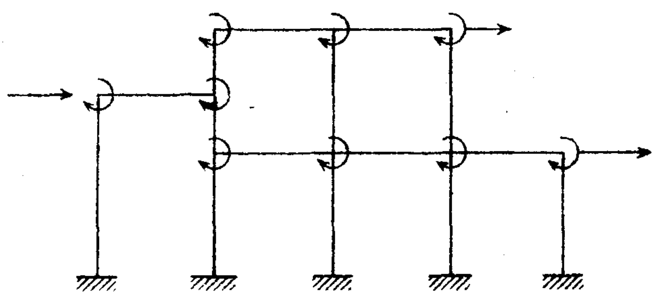
FIG. 3.—INDETERMINATE TRUSS ANALYSIS

dimensional frames are to be analyzed, the torsion of the building, as well as axial deformation of the columns may be included. The computed answer in this type of problem gives the rotation and translation of each end of the columns and beams, and the moments in the members.

Miscellaneous Structures.—The arch rib structure, Fig. 5(a), may be represented by a series of straight beam segments between the load points. The distortion and loading of the structure can then be represented by a system of coordinates (Fig. 5(b)). Twenty-one coordinates are required for a stiffness

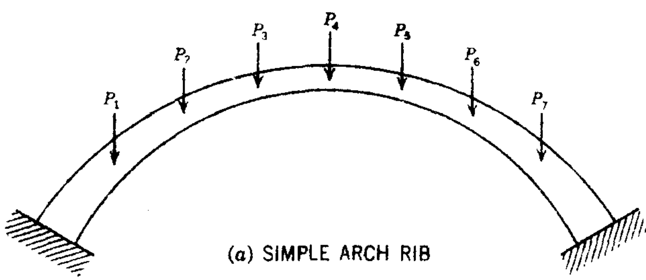


(*a*) SIMPLE BUILDING FRAME

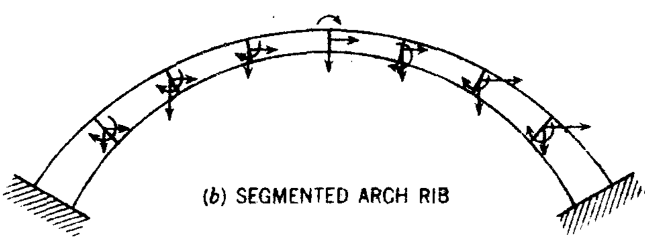


(*b*) BROKEN STORY BUILDING FRAME

FIG. 4.—BUILDING FRAME ANALYSIS



(*a*) SIMPLE ARCH RIB



(*b*) SEGMENTED ARCH RIB

FIG. 5.—ARCH RIB ANALYSIS

matrix analysis. Moments, shears, and axial thrusts are computed at each end of each segment. The stiffness matrix technique can also be used in the analysis of arch dams, suspension bridges, towers, and shell-type structures. A